On complex numbers I – Part 1

Introducing complex numbers, graphing complex numbers, doing arithmetic on complex numbers, and the polar form of a complex numbers.

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1.1 A Brief history of complex numbers

This section is taken from Stuart Dagger at Oxford University: “You met numbers so early in your school career that you will have long since ceased to regard them as anything other than a natural phenomenon, things which are just there and which mathematics uses. This shows how effective early indoctrination can be, for the truth is very different. The positive integers, the numbers that you count with, are natural; the rest are things that mathematicians have invented as an aid to solving problems. As a result, the notion of “number” is one that has changed down the centuries.

Were you to go back to the sixteenth century and the early years of this university, and were you to ask the then professor of mathematics to solve the equation $x^2 + x = 6$, you would be told that the answer was x = 2. Solve the equation yourself and you would get two solutions: $x = 2$ and $x = -3$. The difference is not one of skill or care but of point of view. To the European mathematician of the sixteenth century numbers were things that you counted with and measured with. Negative apples, negative lengths, negative areas didn't exist, and neither, therefore, did negative numbers. The more adventurous were prepared to manipulate negative quantities when doing addition and subtraction, but they weren't prepared to accept negative answers. There is nothing illogical in this; it is just a point of view. Run the clock back further and you will reach a time when there was no such number as zero, and once again this is not illogical; it was just that the need for such a number had not been felt, and so no one had thought to invent it. Notions change as people realise that the old ones are not adequate for their needs.

By the seventeenth century the convenience of being able to manipulate negative quantities was producing such a change of notion. The idea of the real line came in, with positive numbers to the right of zero and negative ones to the left. Interpretations of numbers in terms of credit and debt also helped, but the status of negative numbers was still not on a par with positive ones. And even in the eighteenth century many writers still rejected the idea that you could multiply two negative numbers together. (Positive times negative makes sense in terms of directed distances on the real line, but the “fact” that negative times negative was positive seemed to many to be too paradoxical to be acceptable.) Nowadays nobody worries. Everyone agrees that mathematics is easier and more convenient with negative numbers than without, and so they are accepted. Ways have been devised to give them a sound, logical basis, […].

The reason for telling you all this is that we are now going to make a further extension of what we mean by "number", and again our reason for doing this is that over a period of time mathematicians came to realise that problem solving was easier with the new numbers than without.

It is again convenient to begin by presenting our sixteenth century mathematician with an equation, and this time we shall present him with two:

(1) $$x^2 + 5 = 2x$$

(2) $$x^3 = 15x + 4$$

The first is a quadratic, and the method for solving these has been known since the time of the Babylonians, around 2000 B.C. Applied to this equation it gives you $x = 1 \pm$, at which point our mathematician would have declared that the equation had no solution. You would probably have been told the same thing at school, and, as with the denial of negative numbers, it is an entirely logical view to take. Nor, this time, is there any obvious reason why it might be more convenient to take a different one.

The second equation he'd have found more interesting. The first important mathematical discovery of the European Renaissance was the method for solving cubic equations. It is messy, which is why you don't learn it and why it had not been found earlier, but it exists. Applied to this equation it produces the answer

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}} \qquad (*)$$

This looks like nonsense, and so the natural conclusion is that this is another equation with no solution. However, inspection shows that the equation has the solution $x = 4$. So somehow the right-hand side of (*) must "equal 4".

Motivated by this people began doodling with square roots of negative numbers to see what would happen if they were to exist. They didn't claim that what they were doing was meaningful, just that it was strange and curious and did seem to give correct answers.

Initially the observation was a nine days wonder, rejected as being just too fanciful, but it was taken up again by Leibniz in the late 17th century, and from then through to about 1800 it became increasingly important as a means of discovering results, results which could then be verified by more orthodox means. [...] So it was with "imaginary numbers" in the 18th century.

Here is what Euler, the greatest mathematician of the 18th century, had to say in his book *Elements of Algebra.*

> *Since all numbers which it is possible to conceive are either greater or less than 0, or are 0 itself, it is evident that we cannot rank the square root of a negative number amongst possible numbers, and we must therefore say that it is an impossible quantity. In this manner we are led to the idea of numbers which from their nature are impossible; and therefore they are usually called "imaginary quantities", because they exist merely in the imagination.*
>
> *All such expressions as* $\sqrt{-1}, \sqrt{-2}, \sqrt{-3}, \sqrt{-4}$, *etc are consequently impossible, or imaginary numbers, since they represent roots of negative quantities; and of such numbers we may truly assert that they are neither nothing, nor greater than nothing, nor less than nothing; which necessarily constitutes them imaginary, or impossible.*
>
> *But notwithstanding this, these numbers present themselves to the mind; they exist in our imagination, and we still have a sufficient idea of them; since we know that by* $\sqrt{-4}$ *is meant a number which, when multiplied by itself, produces –4; for this reason also, nothing prevents us making use of these imaginary numbers, and employing them in calculation.*

Having thus asserted the virtues of both practicality and the imagination, Euler went on to explain how to use these imaginary quantities and to show what could be achieved with them. By the early nineteenth century the argument was over: the usefulness of complex numbers was not in dispute, and a means had been found of constructing a sound, logical basis for them. They then took their place as bona fide numbers, much as zero, and negative numbers had earlier."

Below is a table summarising problems and ideas of certain classes of numbers:

	Negative numbers	Real numbers	Complex numbers
Invented to answer things like …	What is 3 – 4?	What is the answer to $x^2 - 2 = 0$	What is $\sqrt{-1}$
Originally not accepted because …	How can you take away more than what you have?	Numbers considered to be expressible in finite terms. $\sqrt{2}$ cannot be a number since its decimal version carries on forever	How can you take the square root of a negative number?
Intuitive meaning	"Opposite" "Mirror" "Reflect along the number line"	$\sqrt{2}$: The hypotenuse of a triangle with unit sides π: the ratio of circumference of a circle to its diameter	"Rotation"

(Adapted from https://betterexplained.com/articles/a-visual-intuitive-guide-to-imaginary-numbers/)

1.2 On quadratic equations having $\Delta < 0$: Defining imaginary and complex numbers

Here we move directly onto defining what a complex number is, along with the most basic situation in which complex numbers arise.

1.2.1 Solving quadratics with $\Delta < 0$ – Part 1

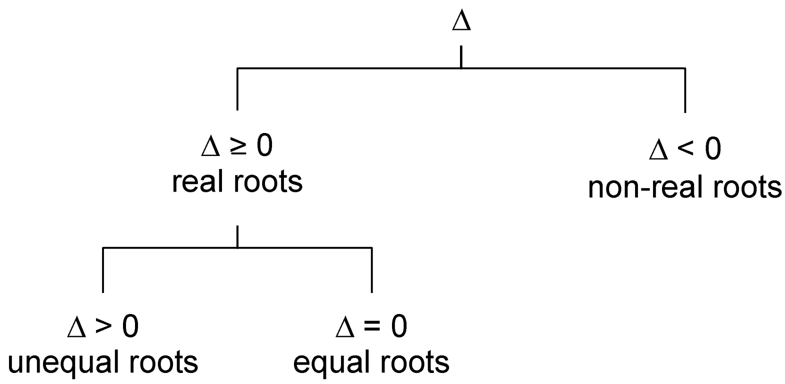
All the algebra we have done in the past has been based on solving equations with real roots. In other words $ax^2 + bx + c = 0$ was solved under the assumption that $b^2 - 4ac \geq 0$. Thus we have been able to solve quadratics such as $x^2 - 2x + 1 = 0$ to give $x = 1, 1$, or $x^2 - 4x + 3 = 0$ to give $x = 1, 3$. However, if we try to solve

$$x^2 - x + 1 = 0$$

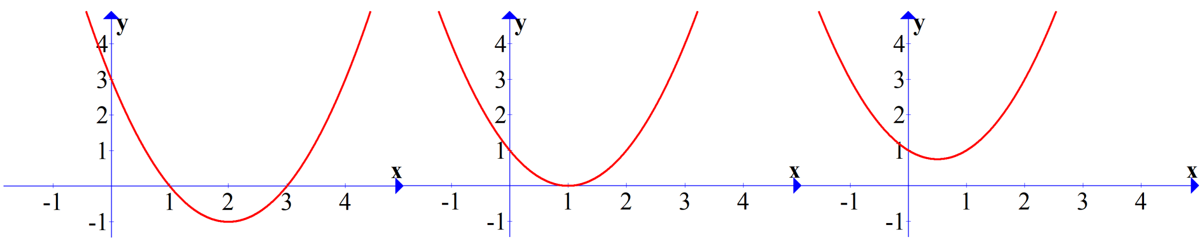
we obtain

$$x = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm \frac{\sqrt{-3}}{2}$$

This result cannot be reduced to real numbers, since $b^2 < 4ac$. Therefore no solution to x exists in $\mathbb{R}$ for this last quadratic. So we see that the solution of a quadratic is dependent on the discriminant $\Delta = b^2 - 4ac$ which adopts one of three possibilities shown below:



The results from the three quadratic equations above can be illustrated graphically as shown below.



$x^2 - 4x + 3 = 0$ $\qquad$ $x^2 - 2x + 1 = 0$ $\qquad$ $x^2 - x + 1 = 0$

1.2.2 Defining the imaginary number

We now come to a critical point. If you have seen, read, or been taught the basics of solving quadratics when $\Delta < 0$ you will have seen the new number i introduced. In introducing this new number you may have been told that it is defined as $i = \sqrt{-1}$. For pratical purpose such a definition is ok, but it can cause confusion as we shall now see (formally speaking (and in terms of modern maths) this is not the correct definition of the imaginary number).

We know that we can take square roots of positive numbers in the following way:

$$\sqrt{4 \times 4} = \sqrt{16} = \pm 4$$

and $$\sqrt{4 \times 4} = \sqrt{4} \times \sqrt{4} = (\pm 2) \times (\pm 2) = \pm 4.$$

In other words, the square root of a product equals the product of the separate square roots. What we are doing here is to distribute the square root across the multiplication, i.e. we are doing $\sqrt{ab} = \sqrt{a}\,\sqrt{b}$.

Will this also be the case when taking roots of complex numbers? Given that we are creating a new number how do we know that it will follow the same rules of arithmetic we use on real numbers? We don't. And, as with all new mathematical objects, we have to go back to the very beginning to define how they work.

So, by not properly understanding the process of rooting complex numbers we can easily fall into the following trap:

$$1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{-1}\,\sqrt{-1} = \left(\sqrt{-1}\right)^2 = -1\,. \qquad (*)$$

So we have "proved" that $1 = -1$. To understand what has gone wrong in our proof let us look at the case of $\sqrt{1}$. The answer to this is

$$\sqrt{1} = \pm 1\,.$$

We know this to be true because we can test both answers, namely

- Squaring the left hand side gives $\left(\sqrt{1}\right)^2 = 1$

and

- Squaring the right hand side gives
  - $(+1)^2 = 1$

and

  - $(-1)^2 = 1$

Hence $\sqrt{1}$ does indeed equal ± 1. So writing $1 = \sqrt{1}$ in (*) above is actually an ambiguous statement, and might even be said to be incorrect, since we have not specified which of the two values of $\sqrt{1}$ we are referring to. The precise way of stating this equation would be as

$$\underset{(a)}{1 = \sqrt{1}} \qquad \text{provided} \qquad \underset{(b)}{\sqrt{1} = +1.}$$

This may seem like a circular statement but it is not since the "$+1$" on the RHS of (b) refers to one of the solutions to $\sqrt{1}$, and the "1" on the LHS of (a) is a number we are claiming is equal to $\sqrt{1}$ (and which is separate to that of the RHS of (b)).

So, if we are interested in showing that $1 = 1$ by using $\sqrt{1}$ we have to choose that answer to the root which makes our maths consistent. In other words we choose $\sqrt{1} = +1$ in order to be able to say $1 = \sqrt{1}$. Similarly, we choose $\sqrt{1} = -1$ in order to say $-1 = \sqrt{1}$.

Note that the solutions $\sqrt{1} = \pm 1$ can be seen to be the result of solving the problem $x = \sqrt{1}$. However, we can restate this problem as that of wanting to solve $x^2 = 1$. It might look as if we are going round in circles, but we are not. The difference between these two equations lies in whether they produce two answers to our problem or only one answer. Although in both cases there are two values of x which satify the equality, the result of solving $x = \sqrt{1}$ is to produce two answers: $x = +1$ and $x = -1$, whereas the result of solving $x^2 = 1$ is to produce only one answer, i.e. the value 1. Conceptually speaking it is the difference between rooting of a number, which produces two different answers, and squaring two different numbers which produces one (and the same) answer.

So, we want square roots not as answers to solving the square root problem but as answers which satisfy the squaring problem, i.e.

$$\sqrt{1} = \pm 1 \textit{ only because } \left(\sqrt{1}\right)^2 = (\pm 1)^2, \text{i.e. } 1 = 1.$$

(if you understand the definition of a function, what we are doing here is to recast the square root problem, which is not a function, as an inverse problem (i.e. as an integer power problem) which is a function).

Hence, defining x to be the solutions such that $x^2 = 1$ instead of $x = \sqrt{1}$ means we no longer need to worry about considering two answers, or which of the two answers to choose. We no longer consider $\sqrt{1} = 1$ and/or $\sqrt{1} = -1$. Instead we consider $(\pm 1)^2 = 1$, i.e. an equation for which it is clear that there is only one answer, namely the value 1

We can now consider the imaginary number i in exactly the same way. The truth is that $\sqrt{-1} = \pm i$. Why? Because if $i = \sqrt{-1}$ then $i^2 = -1$, and if if $-i = \sqrt{-1}$ then $(-i)^2 = -1$. So, when we write

$$1 = \sqrt{(-1)(-1)} = \sqrt{-1} \times \sqrt{-1} = i.i = i^2 = -1$$

we are effectively choosing only one of the roots of $\sqrt{-1}$, i.e. $\sqrt{-1} = i$, and in doing so we have created a contradiction. But we can also chose $\sqrt{-1} = -i$. In that case we have

$$1 = \sqrt{(-1)(-1)} = \sqrt{-1} \times \sqrt{-1} = -i.i = -i^2 = 1$$

which is indeed true. So in order to get the correct answer we have to do is to choose the combination of roots which makes our mathematics consistent. In order that $1 = \sqrt{(-1)(-1)}$ be true we choose the combination $i(-i)$ and we have $1 = i(-i) = -i^2 = 1$.

So, if we are interested in showing that $1 = 1$ via the use of $\sqrt{(-1)(-1)}$ we have to choose that answer to the root which makes our maths consistent. In other words we choose $\sqrt{-1} = i$ as one root and $\sqrt{-1} = -i$ as the other root.

$$1 = \sqrt{(-1)(-1)} = \begin{cases} i.i = i^2 = -1 & \times \\ -i.i = -i^2 = 1 & \checkmark \\ i(-i) = -i^2 = 1 & \checkmark \\ (-i)(-i) = i^2 = -1 & \times \end{cases}$$

Four possible ways of combining the roots i and $-i$,

only two of which give the correct result

The moral of the story is that to save us from the problems of having two answers, and of having to decide which answer to choose, we don't define the imaginary number as $i = \sqrt{-1}$ (which is, in fact, just one of two answers to $\sqrt{-1}$). Rather we define the imaginary number *i* to be a number such that $i^2 = -1$, since it is clear that there is only one answer to i^2. We do this because $\sqrt{(-1)(-1)}$ does not define an answer uniquely equal to 1, whereas the product $(-1)(-1)$ does indeed defines a unique answer 1. Squaring defines unique answers; square rooting does not.

Anyway, it comes to pass that the imaginary number is not defined as $i = \sqrt{-1}$, since this would define *i* to only be the positive square root of -1. Instead, a better way to define it is to say that it is that number which when squared equals -1, i.e.

$$i^2 = -1\,. \qquad (1)$$

Technically speaking, even this is not the actual definition of the imaginary number. The formally accepted defintion of complex numbers refers to something called an ordered pair of numbers which satisfies certain properties of addition and multiplication. But since this takes us way outside the scope of these notes we will take (1) as our working definition.

On a more psychological note, the fact that *i* is called the imaginary number is unfortunate since in the world of modern maths *i* is seen as being as "real" as any other type of number. It may seem to you that *i* is not really a number at all. This attitude is only because we are so used to things such as 1, –2, 3.5 and π as being the "real" numbers, the numbers which really exists.

So when we meet something totally strange like $\sqrt{-1}$ we can't accept it as a number at all. This is exactly the same issue the mathematicians of centuries past faced. But if you continue in your mathematical career you will learn that mathematicians construct many new mathematical objects, one type of which are numbers, and do so for the purpose of being able to solve problems which cannot be solved without them. So from now on we will say that the number *i* is as "real" a number as 1, –2, 3.5 and π, or more precisely that there is does exists a number which, when squared, gives -1. This we can do because i) such a number has been constructed to work as a number according to rules of arithmetic which we will develop later, and ii) it has been possible to construct a whole coherent and consistent framework of mathematics based on the number *i*, and iii) it help solves problems which can't be solved without it.

1.2.3 Solving quadratics with Δ < 0 – *Part 2*

We can now return to our quadratic

$$x^2 - x + 1 = 0$$

of section 1.2.1 whose roots we found to be

$$x = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm \frac{\sqrt{-3}}{2}.$$

The two roots can be expressed as

$$x = \frac{1}{2} \pm i\frac{\sqrt{3}}{2} \tag{2}$$

Equation (2) is then the solution to $x^2 + x + 1 = 0$, and represents two *complex numbers*.

In introducing this new number *i* we have effectively expanded the set of available numbers we can work with, from the set of real number $\mathbb{R}$ to a new set called *the set of complex number*, denoted by $\mathbb{C}$. The set $\mathbb{C}$ is now a set which contains all the other sets of numbers, i.e.

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.$$

We can now solve any and all polynomials, and they can now all be factorised. For the quadratic above we can now say that

$$x^2 + x + 1 = \left(x - \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\right)\left(x - \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\right)$$

Another example: if $x^2 + 25 = 0$ then $x^2 = -25$ which leads to $x = \pm 5i$. Hence we have $(x - 5i)(x + 5i) = 0$. Or if $2x^2 + x + 1 = 0$ then by the quadratic formula we have

$$x = \frac{-1 \pm \sqrt{1 - 4 \times 2 \times 1}}{4} = -\frac{1}{4} \pm \frac{\sqrt{7}}{4} i\,,$$

which can be factorised in the usual way (left as an exercise). We can even solve quadratic whose coefficients are complex numbers (see later).

1.2.4 Defining a complex number

We are now in a position to state the general definition of a complex number as

$$z = a + ib$$

where $a, b \in \mathbb{R}$. Note that both a and b are real numbers, but that a is called the real part (Re) of the complex number z, and b is called the imaginary part (Im) of the complex number z (note that this definition is an informal one. There is actually a much more formal definition, involving something called ordered pairs, which is beyond the scope of these notes).

Examples of complex numbers, along with their Re and Im parts, are shown in the table below.

$\boldsymbol{z = a + ib}$	$2 + 3i$	$-1 - i\pi$	$10i$	3	0
$\boldsymbol{Re(z)}$	2	-1	0	3	0
$\boldsymbol{Im(z)}$	3	$-\pi$	10	0	0

In terms of a quadratic polynomial $y = f(x)$ having complex roots, note that complex numbers always come in pairs. This is because the quadratic formula contains a $\pm$ sign. Hence given one complex root you automatically know the other complex root: if $x_1 = a + ib$ is one root of a quadratic then we know that $x_2 = a - ib$ is automatically the other root.

So, given that complex roots always occur in pairs (but only for polynomials having real coefficients), we can classify the type of roots of a polynomial as follows:

Polynomial	**Type of roots**
Quadratic	Two real root or two complex roots.
Cubic	Three real roots, or one real root and two complex roots.

Polynomial	Type of roots
Quartic	Four real roots, or two real root and two complex roots.
Quintic	Five real roots, three real roots and two complex roots, or one real root and four complex roots.

etc.

Example 1: If we know that a quadratic equation has a root of $2i$ then we also know that its other root is $-2i$. From this we can actually find the quadratic itself since we now the factors of the quadratic to be $x - 2i$ and $x + 2i$. Therefore

$$\begin{aligned}(x-2i)(x+2i) &= x^2 - x[(2i)+(-2i)] + 4\,, \\ &= x^2 + 4\,.\end{aligned}$$

Example 2: Similarly, for a quadratic having roots $2 - i$ and $2 + i$ we can find the quadratic which has these roots as follows:

$$\begin{aligned}\big(x-(2+i)\big)\big(x-(2-i)\big) &= x^2 - x[(2+i)+(2-i)] + (2+i)(2-i)\,, \\ &= x^2 - 4x + 5\,.\end{aligned}$$

Example 3: For a cubic having roots 1, $4 - 3i$ and $4 + 3i$ the cubic can be factorised as

$$(x-1)\big(x-(4+3i)\big)\big(x-(4-3i)\big) = 0\,.$$

For the quadratic part we have

$$\begin{aligned}\big(x-(4+3i)\big)\big(x-(4-3i)\big) &= x^2 - x[(4+3i)+(4-3i)] + (4+3i)(4-3i)\,, \\ &= x^2 - 8x + 25\,.\end{aligned}$$

Hence the actual cubic is $(x-1)(x^2-8x+25) = x^3 - 9x^2 + 33x - 25 = 0$.

Example 4: Given that the sum of two numbers is 4 and their product is 8 then we can set up the system $x + y = 4$ and $xy = 8$, where x and y are the two numbers required. Since the former equation can be expressed as $x = 4 - y$ the latter equation becomes $y(4 - y) = 8$ which simplifies to $y^2 - 4y + 8 = 0$. Here $\Delta < 0$ so we have

$$y = \frac{4 \pm \sqrt{16-32}}{2} = 2 \pm 2i\,.$$

If $y = 2 + 2i$ then $x = 2 - 2i$, and if $y = 2 - 2i$ then $x = 2 + 2i$, hence our two numbers are $x = 2 + 2i$ and $y = 2 - 2i$ (or $y = 2 + 2i$ and $x = 2 - 2i$)

Example 5: A number is called an *algebraic number* if it is the solution to a polynomial equation

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0$$

where $a_0, a_1, \dots, a_n$ are integers. By this definition we can show that $\sqrt{3} + \sqrt{2}$ is an algebraic number as follows: let $z = \sqrt{3} + \sqrt{2}$. Then $z - \sqrt{2} = \sqrt{3}$ (expressing z in this form will ultimately help us eliminate all square roots). Hence

$$\begin{aligned} \left(z - \sqrt{2}\right)^2 &= 3\,, \\ \Rightarrow \quad z^2 - 2\sqrt{2}.z + 2 &= 3\,. \\ \Rightarrow \quad z^2 - 1 &= 2\sqrt{2}.z \end{aligned}$$

Squaring again:

$$\begin{aligned} (z^2 - 1)^2 &= 8z^2\,, \\ \Rightarrow \quad z^4 - 2z^2 + 1 &= 8z^2\,. \end{aligned}$$

Hence

$$z^4 - 10z^2 + 1 = 0$$

is the polynomial having integer coefficients which satisfies $z = \sqrt{3} + \sqrt{2}$. In fact, the number $z = \sqrt{3} + \sqrt{2}$ is a root of this polynomial.

Exercise: Show that $\sqrt[3]{4} - 2i$, $\sqrt[3]{2} + \sqrt{3}$, and $2 - i\sqrt{2}$ are algebraic numbers.

Let us return to the problem of solving $x^3 = 15x + 2$ mentioned in section 1.1 above. As before the following is taken from Stuart Dagger at Oxford University: "This example explains the curious fact we noted earlier in connection with the equation $x^3 = 15x + 2$.

$$\begin{aligned} (2 + i)^3 &= (2 + i)^2(2 + i) \\ &= (3 + 4i)(2 + i) \\ &= 2 + 11i \end{aligned}$$

Look a bit harder at this and you will see the explanation for the problem we hit with the solution to $x^3 = 15x + 2$. We had then a complicated expression involving, among other things,

$\sqrt{-121}$. With our new [complex] numbers $\sqrt{-121}$ can be written as $11i$, since $\sqrt{-121} = \sqrt{121(-1)} = 11\sqrt{-1}$. So $-2 + \sqrt{-121}$ becomes 2 + 11*i*. We have just shown that the cube of 2 + *i* is 2 + 11*i*, and so the cube root of 2 + 11*i* is 2 + *i*. Now verify that $(2 - i)^3 = 2 - 11i$ and note that this tells us that the cube root of 2 − 11*i* is 2 − *i*. Put these discoveries into the right-hand side of the equation (*) we had earlier, and you get *x* = (2 + *i*) + (2 − *i*). So *x* = 4, as required. This is the calculation that intrigued Bombelli in 1572 and acted as the spark for complex numbers."

Examples

1) To solve $x^3 + 3x^2 + 5x + 3 = 0$ we proceed as follows: by trial-and-error (more properly by the factor theorem) we see that $x = -1$ solves this cubic. Hence we have

$$(x+1)(x^2 + bx + c) = x^3 + 3x^2 + 5x + 3 = 0\,.$$

Comparing coefficients we have

$$x^3: \quad 1 = 1 \quad ; \quad x^2: \quad b + 1 = 3$$

$$x: \quad c + b = 5 \quad ; \quad \text{Constant:} \quad c = 3$$

Hence the quadratic part of the cubic is $x^2 + 2x + 3$ which solves as

$$x = \frac{-2 \pm \sqrt{4 - 12}}{2} = -1 \pm i\sqrt{2}\,.$$

Hence the roots of the cubic are $x = -1\,, -1 + i\sqrt{2}\,, -1 - i\sqrt{2}\,.$

Exercise: Find all the roots of $z^2 + (1 + i)z + 5i = 0$.

2) To solve $x^3 - 1 = 0$ we know that $x = 1$ is a root. The other two roots are found as follows:

$$x^3 - 1 = (x - 1)(x^2 + bx + c) = 0\,,$$

after which we may compar coefficients:

$$x^3: \quad 1 = 1 \quad ; \quad x^2: \quad b - 1 = 0$$

$$x: \quad c - b = 0 \quad ; \quad \text{Constant:} \quad -c = -1$$

Hence we have

$$x^3 - 1 = (x - 1)(x^2 + x + 1) = 0\,,$$

from which the roots of the quadratic are

$$x = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}\,.$$

Hence we see that the cubic really does have three roots, two of which are complex. Also, remember that for any polynomial with real coefficients the sum of the roots equals $-b/a$, where a is the coefficient of x^n and b is the coefficient of x^{n-1}. In the case of $x^3 - 1 = 0$ we have $a = 1$ and $b = 0$, so the sum of the roots should be 0. Is this true for our examples? Summing our three roots we have

$$1 + \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) + \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = 1 - \frac{1}{2} - \frac{1}{2} = 0\,,$$

as required. This is an example of a wider range of examples called roots of unity, which we will study in more detail later.

3) Suppose we have the quadratic $x^2 + (20 + 10i)x + w = 0$. How can we find w if this quadratic has a double root? Well, start by setting up the factorised form of a quadratic having a double root, using $a + ib$ as our root. Hence

$$\big(x - (a + ib)\big)^2 = x^2 + (20 + 10i)x + w = 0\,.$$

Expanding the left hand side and comparing coefficients we get

a) for x^2: $-2(a + ib) = 20 + 10i$, implying $-2a = 20$, hence $a = -10$, and $-2b = 10$, hence $b = -5$;
b) for x: $a^2 - b^2 + 2iab = w$, implying from a) that $w = 75 + 100i$.

In other words we have $x^2 + (20 + 10i)x + w = x^2 + (20 + 10i)x + 75 + 100i$, which can be expressed as $\big(x - (-10 - 5i)\big)^2 = 0$.

Exercise: For what real values a, b does $x^2 + (a + 4i)x + b + 24i = 0$ have complex double roots?

4) Given that $z_1 = 1 + 2i$ is a root of $z^2 + az + b = 0$, where $a, b \in \mathbb{R}$, we can can find a and b as follows: Since z_1 is a root we have

$$\begin{aligned} & (1 + 2i)^2 + a(1 + 2i) + b &=& \; 0 \\ \Longrightarrow \quad & (-3 + a + b) + i(4 + 2a) &=& \; 0 \end{aligned}$$

Comparing *Re* and *Im* parts we have

$$\begin{aligned} Im:& \quad 4 + 2a = 0 & \Longrightarrow \quad & a = -2 \\ Re:& \quad -3 + a + b = 0 & \Longrightarrow \quad & b = 5 \end{aligned}$$

Therefore $z^2 - 2z + 5 = 0$. We can test this via the quadratic formula to see if we obtain z_1 as a root.

An alternative solution is to solve the quadratic as follows:

$$z = -\frac{a + \sqrt{a^2 - 4b}}{2} = 1 + 2i\,.$$

Comparing the *Re* part we have we have $-a/2 = 1$, implying $a = -2$. Now, remember that a and b are real, so in order to compare the *Im* part we have to convert $\sqrt{a^2 - 4b}$ to an imaginary number. We can do this as follows: $\sqrt{a^2 - 4b} = \sqrt{(-1)(4b - a^2)} = i\sqrt{4b - a^2}$. Hence

$$\begin{aligned} \frac{\sqrt{4b - a^2}}{2} &= 2 \\ \Longrightarrow \quad 4b - a^2 &= 16 \end{aligned}$$

from which $b = 5$.

1.2.5 Deriving the quadratic formula using complex numbers

This section is adapted from "Discussions: Relating to solutions of quadratic equations", G. R. Dean, *The American Mathematical Monthly*, Vol 22, No. 7. (1915), pp. 243-244.

It is possible to derive the quadratic formula without using the standard approach of factorisation or completing the squre. So, given $i^2 = -1$ let us substitute $x = u + iv$ into the standard form quadratic $ax^2 + bx + c = 0$, where $a, b, c, \in \mathbb{R}$. This gives us

$$a(u + iv)^2 + b(u + iv) + c = 0\,.$$

Expanding and collecting real and imaginary parts we obtain

$$a(u^2 - v^2) + bu + i(2auv + bv) = 0\,.$$

Since the RHS of the above equation can be written as $0 + 0i$, comparing real and imaginary parts LHS and RHS gives us

$$\text{i)} \quad a(u^2 - v^2) + bu + c = 0, \quad \text{and} \quad \text{ii)} \quad v(2au + b) = 0\,.$$

We want to solve these two equation for u and v in terms of a, b, c. Let us consider ii): In general $v \neq 0$ therefore

$$u = -\frac{b}{2a}\,.$$

Substituting this back into i) we obtain

$$\frac{b^2}{4a} - av^2 - \frac{b^2}{2a} + c = 0 \,.$$

Solving this last equation for v gives

$$v = \pm\sqrt{\frac{-b^2}{4a^2} + \frac{c}{a}} = \pm\sqrt{\frac{-b^2 + 4ac}{4a^2}} = \pm\frac{1}{2a}\sqrt{-b^2 + 4ac} \,.$$

Hence we roots

$$x = u + iv = -\frac{b}{2a} \pm i\frac{1}{2a}\sqrt{-b^2 + 4ac} \,.$$

We can now highlight the following from this last equation:

- If $-b^2 + 4ac \geq 0$ then our roots are automatically expressed as complex numbers;
- If $-b^2 + 4ac < 0$ then our roots are real since

$$-b^2 + 4ac < 0 \Longrightarrow -(b^2 - 4ac) = i^2(b^2 - 4ac) > 0$$

  hence

$$x = -\frac{b}{2a} \pm i\frac{1}{2a}\sqrt{i^2(b^2 - 4ac)} = -\frac{b}{2a} \pm \frac{1}{2a}\sqrt{b^2 - 4ac} \,,$$

  which is the standard quadratic formula.

1.2.6 The arithmetic of i

Let us return to the imaginary number *i*. By the fact that $i^2 = -1$ we are able to easily evaluate integer powers of *i*. So, starting with *n* = 0, where *n* is an integer, we have the following first four value of i^n:

i^0	i^1	i^2	i^3
1	*i*	–1	–*i*

The next four powers of *i* are

i^4	i^5	i^6	i^7
1	*i*	–1	–*i*

and the cycle of 1, *i*, –1, –*i* repeats every fourth power. Knowing this we can simplify any integer power of *i*. For example $i^8 = (i^2)^4 = (-1)^4 = 1$, or $i^8 = (i^4)^2 = (1)^2 = 1$. Similarly, $i^{11} = i^{8+3} = i^8 . i^3 = (i^4)^2 . i^2 . i = (1)(-1)i = i$. And another: $i^{105} = i^{104} . i = (i^4)^{26} . i = i$.

In general, when $k = 0, 1, 2, 3, \ldots$ we have the following formulae for the cyclic effect of multiplying by *i*:

$$i^{4k} = 1 , \qquad i^{4k+1} = i , \qquad i^{4k+2} = -1 , \qquad i^{4k+3} = -i .$$

The discussion above related to the multiplicative effect of *i*. There is also a division effect of *i*. As such consider trying to perform $1/i$. In this case we simply multiply this fraction by i/i. Hence

$$\frac{1}{i} = \frac{1}{i}.\frac{i}{i} = \frac{i}{i^2} = -i .$$

We know directly that

$$\frac{1}{i^2} = -1 \qquad \text{and} \qquad \frac{1}{i^4} = 1 ,$$

and for $1/i^3$ we have

$$\frac{1}{i^3} = \frac{1}{i^2}.\frac{1}{i} = i .$$

Rewriting division as multiplication by a negative exponent we can form a table of the cyclic effect of multiplication and division by *i* as shown below:

i^{-4}	i^{-3}	i^{-2}	i^{-1}	i^0	i^1	i^2	i^3
1	i	-1	$-i$	1	i	-1	$-i$

Here we have seen the algebraic effect of multiplying and dividing by *i*. Later on we will see their geometric effect, along with the geometric effect of multiplying and dividing complex numbers by other complex numbers. Now, remember that the geometric effect of multiplying any number by a real number is to strectch or shrink the former number: 2×3 stretches 3 to become 6, ½×6 shrinks 6 to become 3. In the case of of complex numbers we will see that multiplication and division give rise not only to stretching but also to rotation.

More examples

1) Write the following in the form $z = a + ib$:

i) $2i^3 - 3i^2 + 5i$, ii) $5/i + 2/i^3 - 20/i^{18}$, iii) $3i^5 - i^4 + 7i^3 - 10i^2$.

Solution

Using the cyclic properties of i, whereby $i^2 = -1$ and $i^4 = 1$, we have

i) $2i^3 - 3i^2 + 5i = 2(i^2).i - 3i^2 + 5i = -3(-1) + 2(-1).i + 5i = 3 + 3i$;

ii) $3i^5 - i^4 + 7i^3 = 3(i^4).i - i^4 + 7(i^2).i = 3i - 1 - 7i = -1 - 4i$;

iii) $\frac{5}{i} + \frac{2}{i^3} - \frac{20}{i^{18}} = -5i + 2i - \frac{20}{(i^4)^4}.\frac{1}{i^2} = 20 - 3i$.

2) Given that $z = x + iy$, find the following:

i) $Re(iz)$, ii) $Im(z/i\,)$, iii) $Im(1 + iz)$.

Solutions

i) $Re(iz) = Re\big(i(x + iy)\big) = Re(ix + i^2y) = Re(-y + ix) = -y$;

ii) $Im(z/i\,) = Im(-iz) = Im\big(-i(x + iy)\big) = Im(-ix - i^2y) = Im(y - ix) = -x$;

iii) $Im(1 + iz) = Im\big(1 + i(x + iy)\big) = Im(1 + ix + i^2y) = Im(1 - y + ix) = x$.

1.2.7 The conjugate of a complex number

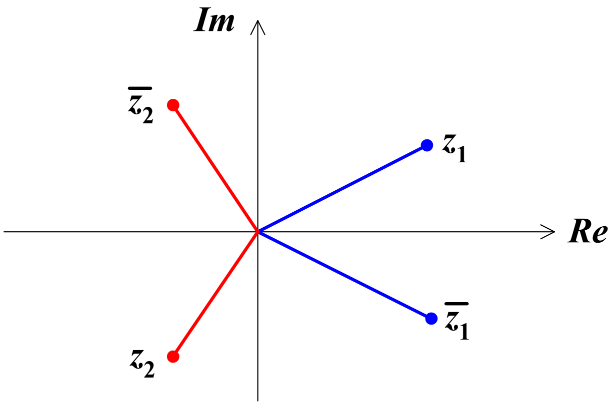
One thing to note about the solutions of all the quadratics above which involved complex roots is that their roots have the same real parts but imaginary parts of opposite sign, i.e. $2 + i$ and $2 - i$, or $4 + 3i$ and $4 - 3i$, etc. This is not a coincidence. This effect comes from the "$\pm$" of the quadratic formula.

In complex number work we therefore define something called the *conjugate* of a complex number. Therefore, given a complex number $z = a + ib$ the conjugate of z, symbolised as z^* or $\bar{z}$, is given by

$$z^* = a - ib \quad \text{or} \quad \bar{z} = a - ib\,.$$

The conjugate is therefore simply a change in the sign of the imaginary part. For example, if $z_1 = 5 + 3i$ then $z_1^* = 5 - 3i$; if $z_2 = -4 - i$ then $z_2^* = -4 + i$. Also note that the conjugate of $5 - 3i$ is $5 + 3i$. In other words, the conjugate of the conjugate leads to the original complex number: $(z_1^*)^* = z_1$.

Geometrically speakng, the conjugate of z is simply a reflection of z about the real axis, as shown below:



If the conjugate is a reflection in the *Re* axis how do we express a reflection in the *Im* axis? We do so as follows: if $z = a + ib$ then $w = -a + ib$ is a reflection in the *Im* axis.

<u>*Example 1:*</u> Given that the complex roots of a quadratic equation with real coefficients occur in conjugate pairs, it is straightforward to find values p and q (where $p, q \in \mathbb{R}$) in $x^2 + px + q = 0$ when one root is known to be i. In that case we know the other root to be $-i$. Hence

$$\begin{aligned} x^2 + px + q &= (x - i)\big(x - (-i)\big)\,, \\ &= x^2 + 1\,. \end{aligned}$$

Comparing coefficients we have $p = 0$ and $q = 1$.

<u>*Example 2:*</u> To find the equation which satisfies $zz^* = 1$ we proceed as follows. If $z = x + iy$ we have $(x + iy)(x - iy) = 1$. This gives $x^2 - ixy + ixy - i^2y = 1$ which simplifies to $x^2 + y^2 = 1$. Hence $zz^* = 1$ can be said to represent a circle of centre $(0, 0)$ and radius 1.

<u>*Example 3:*</u> To find the equation which satisfies $z{+}iz^* + 1 + i = 0$ we proceed as follows. Again, let $z = x + iy$. Then we have

$$(x + iy) + i(x - iy) + 1 + i = 1 + x + y + i(1 + x + y) = 0\,.$$

We now compare *Re* and *Im* parts left and right of this equation to obtain

$$Re\colon 1 + x + y = 0 \text{ and } Im\colon 1 + x + y = 0.$$

Both of these equation imply $y = -1 - x$, therefore $z{+}iz^* + 1 + i = 0$ can be said to represent a straight line of slope -1 and y-intercept -1.

<u>*Exercise*</u>: Find the equation which satisfies i) $z + z^* + 2 = 0$, ii) $z + z^* + 2i = 0$.

Example 4: Given that $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ show that $z_1\overline{z_2} + z_2\overline{z_1}$ is real.

Solution:

$$\begin{aligned} z_1\overline{z_2} + z_2\overline{z_1} &= (x_1 + iy_1)(x_2 - iy_2) + (x_2 + iy_2)(x_1 - iy_1)\,, \\ &= x_1x_2 + y_1y_2 + i(x_2y_1 - x_1y_2) + x_1x_2 + y_1y_2 - i(x_2y_1 - x_1y_2)\,, \\ &= 2(x_1x_2 + y_1y_2)\,. \end{aligned}$$

Hence $z_1\overline{z_2} + z_2\overline{z_1}$ is real.

Example 5: Write the following in the form $z = x + iy$:

i) $z - 2z^* + 7 - 6i = 0$, ii) $\bar{z} = 4z$, iii) $2z = i^*(2 + 9i)$.

Solutions

i) $z - 2z^* + 7 - 6i = x + iy - 2(x - iy) + 7 - 6i = (-x + 7) + i(3y - 6) = 0$;

ii) $\bar{z} = x - iy$, and $4z = 4(x + iy) = 4x + 4iy$. Hence

$$x - iy = 4x + 4iy\,,$$

$$\Longrightarrow \qquad 3x + 5iy = 0\,.$$

iii) $2z = 2(x + iy) = 2x + 2iy$, and $i^*(2 + 9i) = -i(2 + 9i) = -2i + 9$. Hence

$$2x + 2iy = 9 - 2i\,,$$

$$\Longrightarrow \qquad (2x - 9) + 2i(y + 1) = 0\,.$$

Example 6: Given that $z = x + iy$, what can be said about $\bar{z} = z$ and $\bar{z} = iz$?

Solution: If $\bar{z} = z$ then $x - iy = x + iy$. Hence $2iy = 0$, i.e. $y = 0$. Therefore z is real.

If $\bar{z} = iz$ then $x - iy = i(x + iy)$. Hence $x - iy = -y + ix$, i.e. $x + y = i(x + y)$. Dividing by $x + y$ we have $i = 1$. But we know that $i = \sqrt{-1}$ hence there is no equation which satisfies $\bar{z} = iz$.

We can now use the definition of the conjugate of $z = x + iy$ to express x and y in terms of z and $\bar{z}$ as follows: $z + \bar{z} = 2x$ and $z - \bar{z} = 2iy$. Therefore

$$Re\,(z) = x = \frac{z + \bar{z}}{2} \quad \text{and} \quad Im(z) = y = \frac{z - \bar{z}}{2i}\,. \tag{3}$$

For example, if $z = -2 + i$ then $\bar{z} = -2 - i$, hence

$$x = \frac{-2 + i - 2 - i}{2} = 4 \qquad \text{and} \qquad y = \frac{-2 + i - (-2 - i)}{2i} = 2i\ .$$

Equations (3) can then be used to express any equation in Cartesian form into one in complex number form. Such a complex number form is called *complex conjugate coordinates*. For example if that $z = x + iy$, then we can express $2x + y = 5$ as

$$2\frac{(z + \bar{z})}{2} + \frac{(z - \bar{z})}{2i} = 5\ .$$

Cross multiplying and expanding we get

$$(2i + 1)z + (2i - 1)\bar{z} = 10i\ .$$

On the other hand we can also find the Cartesian equation of an expression already expressed in complex-conjugate form. For example, if $z + \bar{z} = 4$ then $x + iy + x - iy = 4$. Implying that $x = 2$.

Examples

1) Given that $z = x + iy$, express each of $mx + ny = k$ (where $m, n \in \mathbb{R}$) , $x^2 + y^2 = 36$ and $(x - 3)^2 + y^2 = 9$ in complex conjugate form.

   Solution

   If $mx + ny = k$ (where $m, n \in \mathbb{R}$) then by (3) we have

$$m\left(\frac{z + \bar{z}}{2}\right) + n\left(\frac{z - \bar{z}}{2i}\right) = k\ .$$

   Cross-multiplying by 2, and remembering that $1/i = -i$, this simplifies to

$$(m - ni)z + (m + ni)\bar{z} = 4k\ .$$

   If $x^2 + y^2 = 36$ then by (3) we have

$$\left(\frac{z + \bar{z}}{2}\right)^2 + \left(\frac{z - \bar{z}}{2i}\right)^2 = 36\ .$$

   Expanding we get

$$\frac{1}{4}(z^2 + 2z.\bar{z} + \bar{z}^2) - \frac{1}{4}(z^2 - 2z.\bar{z} + \bar{z}^2) = 36\ ,$$

   from which we obtain

$$z.\bar{z} = 36\ .$$

Another way of solvint his is to notice that $z.\bar{z} = (x+iy)(x-iy) = x^2 + y^2$. Hence we have directly that $z.\bar{z} = 36$.

If $(x-3)^2 + y^2 = 9$ then we have $x^2 - 6x + 9 + y^2 = 9$. Then by (3) we obtain

$$\left(\frac{z+\bar{z}}{2}\right)^2 - 6\left(\frac{z+\bar{z}}{2}\right) + 9 + \left(\frac{z-\bar{z}}{2i}\right)^2 = 9\,.$$

Expanding gives

$$\frac{1}{4}(z^2 + 2z.\bar{z} + \bar{z}^2) - 3(z+\bar{z}) + 9 - \frac{1}{4}(z^2 - 2z.\bar{z} + \bar{z}^2) = 9\,,$$

from which we obtain

$$z.\bar{z} = 3(z+\bar{z})\,.$$

2) Left as an exercise: Given that $z = x + iy$, express $(x-a)^2 + (y-b)^2 = r^2$ in complex conjugate form, where $a, b \in \mathbb{R}$.

3) Find the Cartesian equations of the complex-conjugate equations $\bar{z} = z + 6i$ and $z\bar{z} - 2z - 2\bar{z} - 8 = 0$.

<u>*Solution*</u>

If $\bar{z} = z + 6i$, then

$$x - iy = x + iy + 6i\,,$$

which simplifies to $y = -3$.

If $z\bar{z} - 2z - 2\bar{z} - 8 = 0$, then

$$(x+iy)(x-iy) - 2[(x+iy) + (x-iy)] - 8 = 0\,,$$

which simplifies to

$$x^2 - 2x + y^2 - 8 = 0\,.$$

This expression can be transformed into the equation of a circle by completing the square on $x^2 - 2x$. Hence

$$x^2 - 2x + 1 - 1 + y^2 = 8\,,$$

which simplifes to

$$(x-1)^2 + y^2 = 9\,.$$

1.2.8 Certain properties of the conjugate

In this section we will go through proofs of certain properties involivng the conjugate of a complex number z. We will start with a few simple properties which may seem obvious. The point about showing these simple properties is to gain experience in the nature and presentation of proofs.

Property 1: If $z = x + iy$ prove $(z^*)^* = z$.

Proof: Given $z = x + iy$ then $z^* = (x + iy)^* = x - iy$. Hence $(z^*)^* = (x - iy)^* = x + iy = z$.

■

Comment: The nature of a proof is that every mathematical statement beyond what is already given, or known prior, has to be explcitely developed. So, since z^* is not given to us we must develop the step which leads to it. Now, this may seem trivial, but it is in the nature of proofs that we should do this explicitly, hence the reason for me writing "$z^* = (x + iy)^* = x - iy$". More than this, I have presented the intermediate step of $(x + iy)^*$ as a matter of clarity (although this need not be done).

I have hten repeated this whole form of presentation for $(z^*)^*$, at the end of which I have explicitely stated the equality with z (as the last statement of the proof) as a matter of definitively confirming the original statement.

Property 2: If $z = x + iy$ prove $z.z^* = |z|^2$.

Proof: Given $z = x + iy$ then $|z|^2 = \left(\sqrt{x^2 + y^2}\right)^2 = x^2 + y^2$. Also, we have $z^* = (x + iy)^* = x - iy$. Therefore $z.z^* = (x + iy)(x - iy) = x^2 + y^2 = |z|^2$. ■

Property 3: If z is such that $|z|^2 = 1$ prove $z^* = 1/z$.

Proof: By property 2 we know $z.z^* = |z|^2$. If $|z|^2 = 1$ then $z.z^* = 1$, implying $z^* = 1/z$. ■

Property 4: If $z = x + iy$ prove $(z^2)^* = (z^*)^2$

Proof: Given $z = x + iy$ then $z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy$. Therefore $(z^2)^* = x^2 - y^2 - 2ixy$. Now, $z^* = x - iy$, hence $(z^*)^2 = (x - iy)^2 = x^2 - y^2 - 2ixy$. Thus $(z^2)^* = (z^*)^2$.

■

It is true that $(z^n)^* = (z^*)^n$ for any integer n. this can be proved by induction, which is left as an exercise.

Property 5: For a complex number z prove $z = z^*$ if and only if z is real.

Proof: i) Given $z = x + iy$ we have $z^* = x - iy$. Then $x + iy = x - iy$ only if $y = 0$ implying that z is real; ii) on the other hand, if z is real then, by definition of $z = x + iy$, we have $y = 0$.

■

As an example, if $z = 7$ then $z^* = 7$.

Property 6: If z_1 and z_2 are two complex numbers prove $(z_1 + z_2)^* = z_1^* + z_2^*$.

Proof: Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then

$$\begin{aligned}
(z_1 + z_2)^* &= (x_1 + iy_1 + x_2 + iy_2)^* , \\
&= \left(x_1 + x_2 + i(y_1 + y_2)\right)^* , \\
&= x_1 + x_2 - i(y_1 + y_2) , \\
&= x_1 - iy_1 + x_2 - iy_2 , \\
&= (x_1 + iy_1)^* + (x_2 + iy_2)^* , \\
&= z_1^* + z_2^* .
\end{aligned}$$

■

By extension we have $(z_1 + z_2 + z_3)^* = (z_1 + z_2)^* + z_3^* = z_1^* + z_2^* + z_3^*$, etc.

Note that a complex number is a number of the form/structure $z = X + iY$. We can only take the conjugate of a complex number when z is of this form. Hence we have had to do some initial algebra in order to transform the two separate complex numbers z_1 and z_2 into the form of the compound complex number "$z_1 + z_2$". Only then can we take the conjugate of this compound number. From this we then separate out the components of z_1 and z_2 in order to finish the proof. As an example, if $z_1 = 1 - 2i$, $z_2 = -2 - i$ and $z_3 = 3 - 6i$, then $z_1 + z_2 + z_3 = 2 - 9i$. Hence $(z_1 + z_2 + z_3)^* = 2 + 9i$. On the other hand, $z_1^* = 1 + 2i$, $z_2 = -2 + i$, and $z_3 = 3 + 6i$. Therefore $z_1^* + z_2^* + z_3^* = 2 + 9i = (z_1 + z_2 + z_3)^*$.

Property 7: If z_1 and z_2 are two complex numbers prove $(z_1 z_2)^* = z_1^* z_2^*$.

Proof: Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then

$$\begin{aligned}
(z_1 z_2)^* &= [(x_1 + iy_1)(x_2 + iy_2)]^* , \\
&= \left(x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1)\right)^* ,
\end{aligned}$$

$$
\begin{aligned}
\text{So } (z_1 z_2)^* &= x_1x_2 - y_1y_2 - i(x_1y_2 + x_2y_1)\,, \\
&= x_1x_2 - i(x_1y_2 + x_2y_1) + i^2 y_1y_2\,, \\
&= (x_1 - iy_1)(x_2 - iy_2)\,, \\
& \ (x_1 + iy_1)^*(x_2 + iy_2)^*\,, \\
&= z_1^* z_2^*\,.
\end{aligned}
$$

■

The same comment applies here as for property 4: a complex number is a number of the form/structure $z = X + iY$. We can only take the conjugate of a complex number when z is of this form. Hence we have had to do some intial algebra in order to transform the two separate complex numbers z_1 and z_2 into the form of the compound complex number "z_1z_2". Only then can we take the conjugate of this compound number. From this we then separate out the components of z_1 and z_2 in order to finish the proof.

As an example, if $z_1 = 1 - 2i$, and $z_2 = -2 - i$ then $z_1z_2 = -1 + 3i$. Hence $(z_1z_2)^* = -4 - 3i$. On the other hand $z_1^* = 1 + 2i$ and $z_2^* = -2 + i$. Hence $z_1^*z_2^* = -4 - 3i = (z_1z_2)^*$.

<u>*Exercise:*</u> Prove $(z_1/z_2)^* = z_1^*/z_2^*$.

<u>*Property 8:*</u> If z_1 and z_2 are two complex numbers prove $z_1z_2^* + z_1^*z_2 = 2Re(z_1z_2^*)$

<u>*Proof:*</u> Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then

$$
\begin{aligned}
z_1z_2^* + z_1^*z_2 &= (x_1 + iy_1)(x_2 + iy_2)^* + (x_1 + iy_1)^*(x_2 + iy_2), \\
&= (x_1 + iy_1)(x_2 - iy_2) + (x_1 - iy_1)(x_2 + iy_2), \\
&= x_1x_2 + y_1y_2 - i(x_1y_2 - x_2y_1) + x_1x_2 + y_1y_2 + i(x_1y_2 - x_2y_1)\,, \\
&= 2(x_1x_2 + y_1y_2)\,.
\end{aligned}
$$

Also,

$$
\begin{aligned}
2Re(z_1z_2^*) &= 2Re[(x_1 + iy_1)(x_2 - iy_2)]\,, \\
& \ 2Re[x_1x_2 + y_1y_2 - i(x_1y_2 - x_2y_1)]\,, \\
&= 2(x_1x_2 + y_1y_2)\,.
\end{aligned}
$$

Hence $z_1z_2^* + z_1^*z_2 = 2Re(z_1z_2^*)$. ■

As an example, if $z_1 = 1-2i$, and $z_2 = -2-i$ then $z_1 z_2^* = (1-2i)(-2+i) = 5i$, and $z_1^* z_2 = (1+2i)(-2-i) = -5i$. Therefore, $z_1 z_2^* + z_1^* z_2 = (5i) + (-5i) = 0$. But note that $2Re(z_1 z_2^*) = 2Re(5i) = 0$. Hence $z_1 z_2^* + z_1^* z_2 = 2Re(z_1 z_2^*)$.

Property 9: If z_1 is a root of $P(z) = az^2 + bz + c$, where $a, b, c \in \mathbb{R}$, show that $\overline{z_1}$ is also a root.

Proof: We know that $P(z_1) = az_1^2 + bz_1 + c = 0$. For $P(\overline{z_1})$ we have

$$P(\overline{z_1}) = a(\overline{z_1})^2 + b(\overline{z_1}) + c\,.$$

By the property $(\bar{z})^n = \overline{(z^n)}$ (left as an exercise) we have

$$P(\overline{z_1}) = a\overline{z_1^2} + b\overline{z_1} + c\,.$$

Since a, b, and c are real numbers they are their own conjugates we can write

$$P(\overline{z_1}) = \overline{az_1^2} + \overline{bz_1} + \bar{c}\,.$$

By the property $\overline{w+z} = \overline{w} + \bar{z}$ we have

$$P(\overline{z_1}) = \overline{az_1^2 + bz_1 + c}\,.$$

But we know that $az_1^2 + bz_1 + c = 0$ hence $P(\overline{z_1}) = 0$.

This property applies to a polynomial of any degree, not just a quadratic. In other words, if z_1 is a root of $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, where $a_0, a_1, \ldots, a_{n-1}, a_n \in \mathbb{R}$, then $\overline{z_1}$ is also a root of $P(z)$.

■

Properties such as those above are useful in simplifying expressions involving complex numbers or functions. For example, consider the complex quadratic $f(z) = 2z^2 - 3z + 1$. Then

$$
\begin{aligned}
\big(f(z)\big)^* &= (2z^2 - 3z + 1)^*\,, \\
&= (2z^2)^* - (3z)^* + (1)^*\,, && \text{by property 2.} \\
&= (2)^*(z^2)^* - (3)^*(z^*) + (1)^* && \text{by property 3.} \\
&= 2(z^*)^2 - 3z^* + 1\,. && \text{by property 4. \& property 6.}
\end{aligned}
$$

Exercises:

Given two complex numbers z and w, where relevant, are the following statements true? If they are prove them, otherwise find a counter example: 1) $(z^* + w)^* = z + w^*$, 2) $z + z^* = 2Re(z)$, 3) $z - z^* = 2i.Im(z)$, 4) $z/z^* = (z + z^*)/|z|^2$.

1.3 Complex roots can be located on a Cartesian graph – Part 1

1.3.1 Locating the complex roots of a quadratic equation.

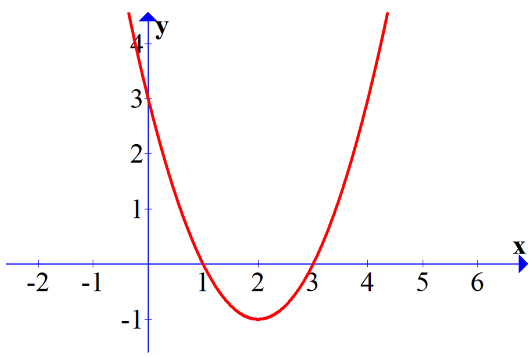
The following idea is based on the paper "74.35 Imagine the Roots of a Quadratic", C. R. Holmes, *The Mathematical Gazette*, Vol. 74, No. 469 (Oct., 1990), pp. 285-286.

Consider the three quadratics and their graphs on the folowing page. Since the roots of a) and b) are real they can be seen on the standard x–y graph as the points where $f(x)$ intersects the x-axis. Therefore we can read off the roots for a) and b) directly from each graph. However, there is another way of reading such roots:

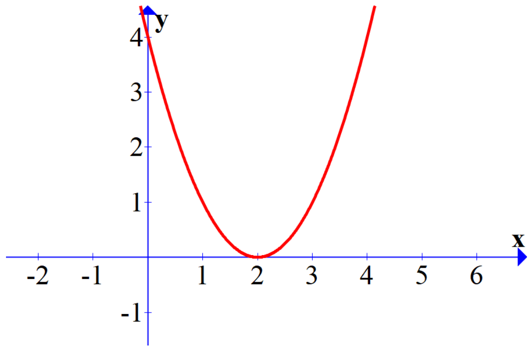
i) Find the x value of the minimum point (algebraically this is the point of symmetry given by $x = -b/(2a)$). In the case of a) we have $x = 2$;

ii) Find the distance d of this x value to the points of intersection of $f(x)$ with the x-axis. In the case of a) we have $d = 1$;

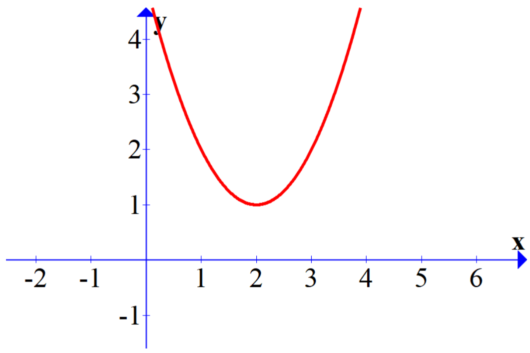
iii) Find the roots by adding ii) to i), and subtracting ii) from i). In the case of a) we have one root $= 2 - 1 = 1$, and the other root $= 2 + 1 = 3$;

	Quadratic	Roots	Graph
a)	$f(x) = x^2 - 4x + 3 = 0\,,$	$x = 1, 3$	

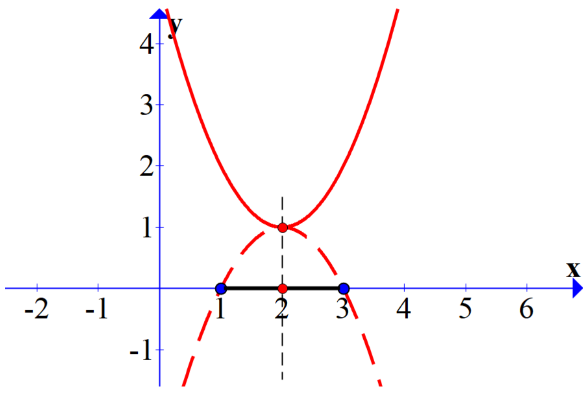
b) $f(x) = x^2 - 4x + 4 = 0$, $\quad x = 2, 2$



c) $f(x) = x^2 - 4x + 5 = 0$, $\quad x = 2 \pm i$



For c) there are no points of intersection since the roots of $f(x)$ are complex, so we cannot read off the roots directly from the graph. However, suppose we reflect the curve shown in c) about its minimum point. Our function becomes $g(x) = -x^2 + 4x - 3$, and we then have the curve shown as a red dash in the graph below:

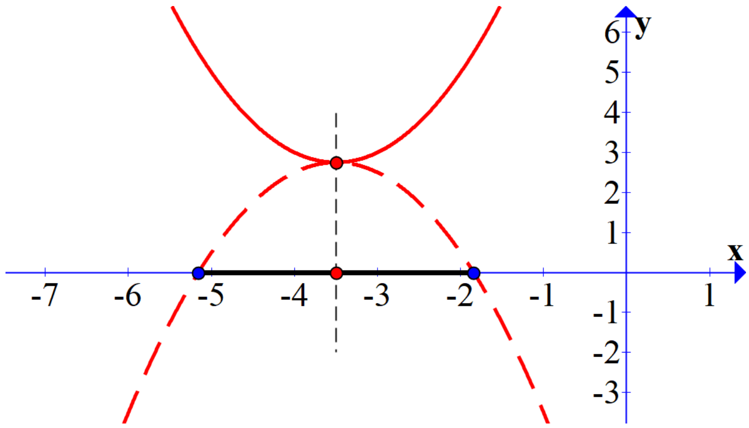


The roots of $g(x)$ are $x = 1, 3$. Then, by using the procedure described in i) – iii) above we can obtain the complex roots as follows:

- the real part of the complex number is the x value of the point of symmetry (now a maximum point). Here $Re(z) = 2$;

- the imaginary part is the distance from this x value to the point of intersection of $f(x)$ with the x-axis. Here $Im(z) = 1$;
- distances to the left of the point of symmetry are negative values, and distances to the right of the point of symmetry are positive values. Hence $z_1 = 2 - i$, and $z_2 = 2 + i$.

Similarly if $f(x) = x^2 + 7x + 15$ we reflect $f(x)$ about the line containing the point of symmetry.



Our reflected function is then

$$g(x) = -x^2 - 7x - 9.5 \,.$$

Then we find the points of intersection of $g(x)$ with the x-axis (i.e. the roots of $g(x)$), these being

$$x_{roots} \approx -5.158, -1.842 \,.$$

Then we find the distance between x_{roots} and $x_{symmetry}$, this being

$$d = 1.658 \,.$$

Noting that the distance to the left of x_{roots} is negative and distance to the right of x_{roots} is positive we have the complex roots of $f(x)$ to be $z_1 = -3.5 + 1.658i$ and $z_2 = -3.5 - 1.658i$.

In general it can be shown that the the real roots of a reflected quadratic lead to the complex root of the original quadratic. To do this we will need to find the equation of the line about which we releflect the quadratic, and then solve this reflected quadratic for its (real) roots.

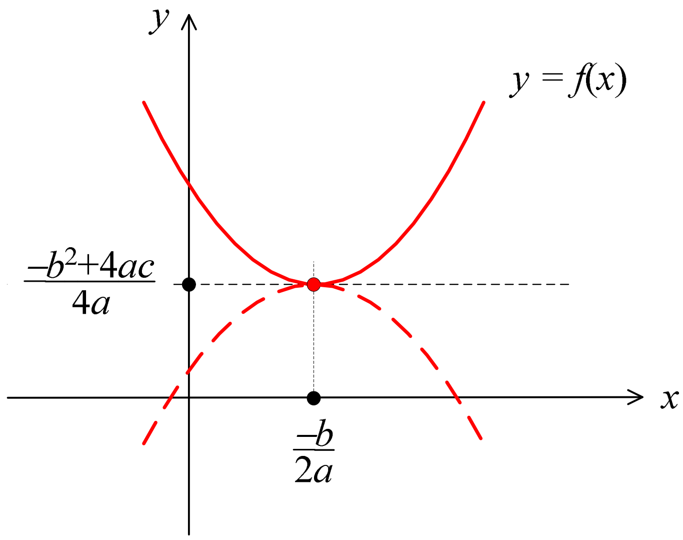
Therefore, let $f(x) = ax^2 + bx + c = 0$ be a quadratic such that $\Delta = b^2 - 4ac < 0$. We know from standard algebra that the x-value of the point of symmetry of $f(x)$ is $x_{symmetry} = -b/(2a)$ (found by the usual procedure of completing the square on $f(x)$).

In order to reflect $f(x)$ about the horizontal line containing $x_{symmetry}$ we need to find the equation of this line. We do this by substituting $x_{symmetry} = -b/(2a)$ into $f(x)$ to get

$$y = f\left(-\frac{b}{2a}\right) \quad = \quad a\left(-\frac{b}{2a}\right)^2 + b\left(-\frac{b}{2a}\right) + c \,,$$

$$\Rightarrow \qquad\qquad y \quad = \quad \frac{-b^2 + 4ac}{4a}.$$

We now have the equation of the line about which we need to reflect $y = f(x)$, as illustrated by the configuration illustrated below:



Now, reflecting the curve of $f(x)$ about a horizontal line $y = K$ is done via the transformation $y = -(f(x) - K) = -f(x) + K$. Let our reflected quadratic be $g(x)$. We then have

$$g(x) = -ax^2 - bx - c + \frac{-b^2 + 4ac}{4a}.$$

By design, $g(x)$ has real roots, so we can solve this as usual by the quadratic formula, viz:

$$x = \frac{b \pm \sqrt{(-b)^2 - 4\left[(-a)\left(-c + \frac{-b^2 + 4ac}{4a}\right)\right]}}{-2a},$$

which, after some algebra, simplifies to

$$x = \frac{b \pm \sqrt{-b^2 + 4ac}}{-2a}. \tag{4}$$

Now notice that the discriminant of (4): this can be written as $\sqrt{(-1)(b^2 - 4ac)} = i\sqrt{b^2 - 4ac}$. Hence the real roots $x_{g(x)}$ of $g(x)$ are the complex roots $x_{f(x)}$ of $f(x)$:

$$x_{g(x)} = -\frac{b \pm \sqrt{-b^2 + 4ac}}{2a} = -\frac{b \pm i\sqrt{b^2 - 4ac}}{2a} = x_{f(x)},$$

when $f(x)$ has discriminant $\Delta < 0$.

1.3.2 Locating the complex roots of a cubic equation.

Just as the complex roots of a quadratic can be found on an x-y graph, so can the complex roots of a cubic. Now, we know that a cubic equation has either three real roots, or one real root and two complex roots. It is this latter form of cubic which we will now study in order to see how the graph of such a cubic shows the location of the complex roots.

As such let the roots of the general cubic $y = px^3 + qx^2 + rx + s = 0$ be $x = c$ and $x = a \pm ib$, where $a, b, c \in \mathbb{R}$. Such a cubic can theefore be factorised as

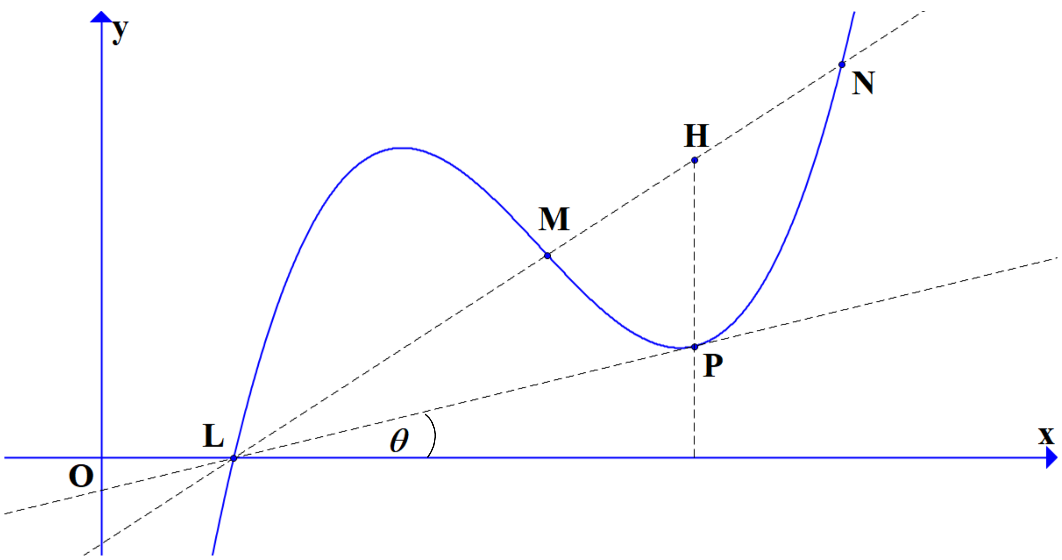
$$y = (x - c)(x - [a + ib])(x - [a - ib]) = 0\,,$$

which simplifies to

$$y = (x - c)(x^2 - 2ax + a^2 + b^2) = 0\,.$$

Our aim is to find a and b, and to do this will will need two equations. We obtain these two equations by using whatever geometric means we can. In this case we will use the simple geometric properties of secants and tangent to the cubic.

The diagram below will help us visualise such geometry as we go through our analysis.



We first draw a secant LMN through the cubic, where L is the real root $x = c$. The general equation of a line is given by $y - y_1 = m(x - x_1)$, hence for line LMN we have

$$y = m(x - c)\,.$$

When this line intersects the cubic at M and N we obtain

$$m(x - c) = (x - c)(x^2 - 2ax + a^2 + b^2)\,,$$

which simplifies to

$$x^2 - 2ax + a^2 + b^2 - m = 0\,.$$

Solving this gives the *x* ordinate of M and N to be

$$x = a \pm \sqrt{m - b^2}\,. \tag{*}$$

Points M and N will have the same x ordinate when the two equations in (*) are equal, implying that $m = b^2$, from which we have $x = a$. The line becomes LP which is now tangent to the cubic. But this tangent is also the slope of line LP. So

$$\text{slope of LP} = b^2 = \tan\theta\,,$$

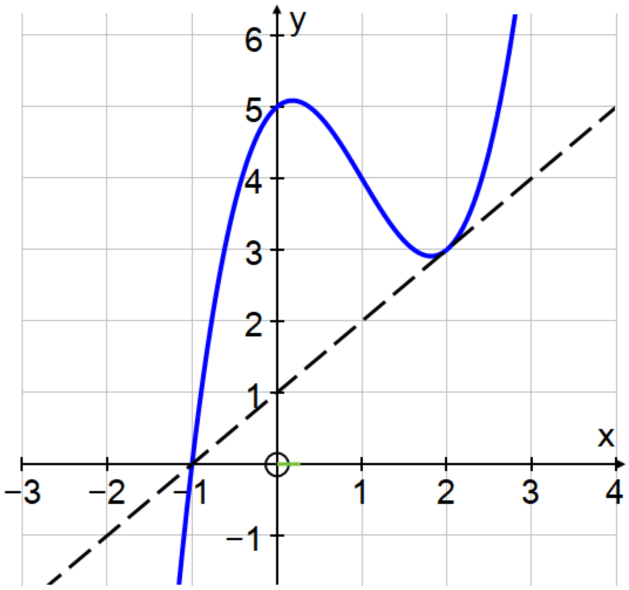
Therefore, our complex roots $x = a \pm ib$ can be located as follows: for the tangent to the cubic through the real root $x = c$ is such that

- the real part a is the *x* ordinate of the point of tangency P. This can be read off directly by drawing a vertial line from P to the *x*-axis;
- the imaginary part b is found as $b = \sqrt{\tan\theta}$, where θ is read off the graph, or by $m = b^2$. In this latter case we find b as $b = \sqrt{|m|}$ since we are only interested in the magnitude of the slope and not its direction (i.e. whether it is positive or negative). See example 3 below for an illutration of this.

Note that the appearance of $\tan\theta$ as part of the analysis above for complex roots is not a coincidence. It plays a fundamental role as we shall see in sections 1.11 onwards.

Example 1:

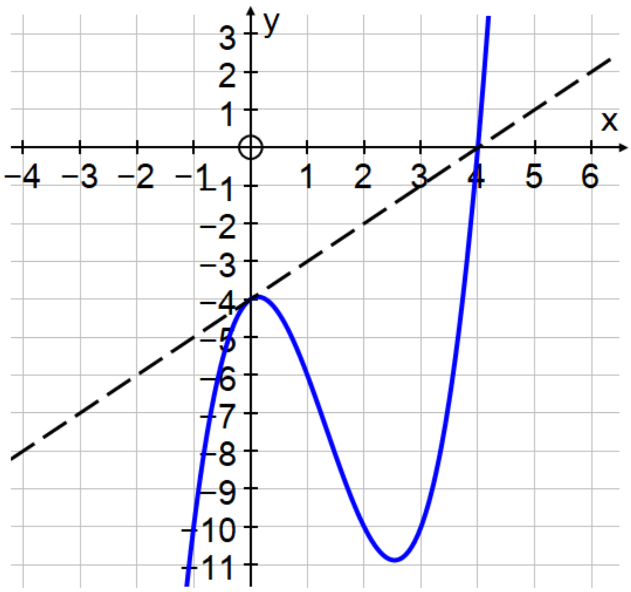
Plotting the cubic $y = x^3 - 3x^2 + x + 5$ we have the graph below. From this we see that there is only one real root of y located at $x = -1$. The other two roots are therefore complex, of the form $x = a \pm ib$. We now draw a line which passes through $(-1, 0)$ which is tangent to the curve, as shown.



From the graph we see that the point of tangency has x ordinate $x = 2$. This value is the real part of the complex root. For the imaginary part we need to find the angle of the tangent line, which can be easly seen to be $\theta = \pi/4$. Hence $b = \sqrt{\tan \pi/4} = \pm 1$. Hence the roots of the cubic are given as $x = -1$, $x = 2 + i$, and $x = 2 - i$.

Example 2:

Plotting the cubic $y = x^3 - 3x^2 + x + 5$ we have the graph below



Again we see that the real root is given by $x = 4$. The line drawn through $x = 4$ which is tangent to the cubic is illustrtaed by the black dashed line.

By visual inspection we see that the x-ordinate of the point of tangency is $x = 0$. Since this represents the real part of the complex root we have that there is no real part to the complex roots of y.

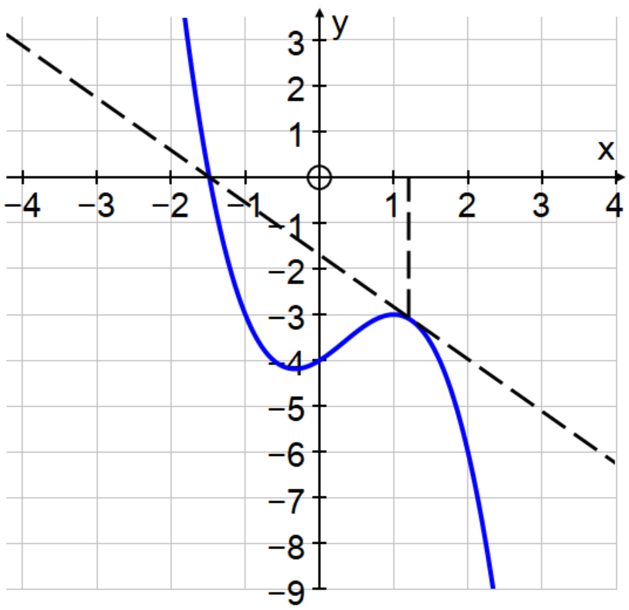
We now find the angle the tangent line makes with the x-axis, which can be seen to be $\theta = \pi/4$. Hence the imaginary parts of the complex roots are $b = \sqrt{\tan \pi/4} = \pm 1$.

Hence the roots of the cubic are given as $x = 4$, $x = i$, and $x = -i$.

<u>*Example 3:*</u>

Plotting the cubic $y = -x^3 + x^2 + x - 4$ we have the graph below. By visual inspection we find that the real root of y is $x \approx -1.5$.

Then, using either the method of bisection or Newton-Raphson method (two examples of numerical methods for finding approximate solutions to roots of equations) we can use this value as the initial estimate x_0 to obtain the improved solution $x \approx -1.4856$ to 4 d.p.



We then draw a line through $x \approx -1.4856$ which is tangent to the cubic as shown by the black dashed line. Again by visual inspection we estimate the x ordinate of the point of tangency to be $x \approx 1.2$ (illustrated by the vertical dashed line from the tangent point to the x-axis). This value is the real part of the complex root. For the imaginary part we need to find the angle of the tangent line.

This can be found as

$$m \approx \frac{-y(1.2)}{1.2-(-1.4856)} = -\frac{3.1}{2.686} = -1.154\,.$$

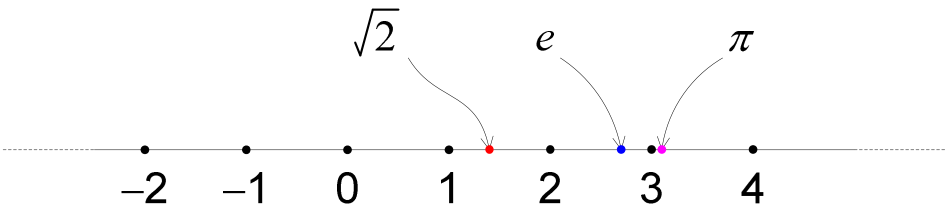
Hence

$$b = \sqrt{|m|} = \sqrt{1.154} = 1.074\,.$$

Hence the roots of the cubic are given approximately as $x \approx 1.2 + 1.074i$, $x \approx 1.2 - 1.074i$, and $x \approx -1.486$

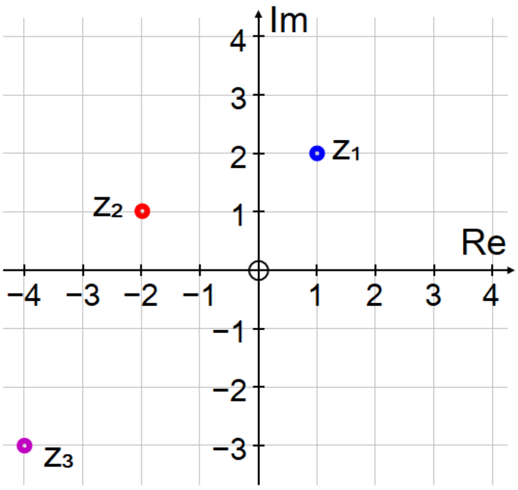
1.4 The Argand diagram, |z| and arg(z)

Recall that real numbers can be represented geometrically as points on a number line:

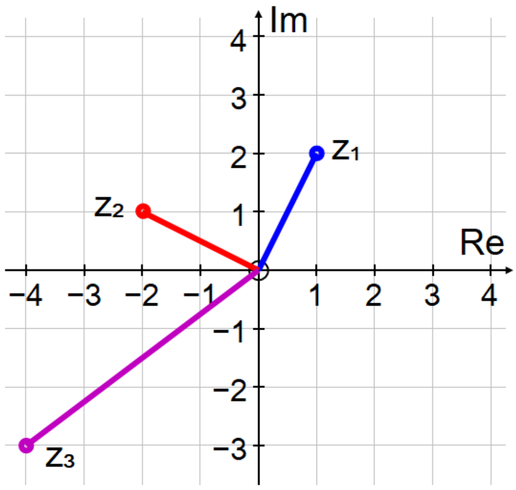


They can also be seen as distances, measured from 0.

Similarly, complex numbers can be represented geometrically. But because they have two components, a real part and an imaginary part, we draw complex numbers as points on a two dimensional graph. We therefore represent complex numbers on a graph by plotting the real part on a horizontal *Re* axis, and the imaginary part on a vertical *Im* axis. So numbers such as $z_1 = 1 + 2i$ or $z_2 = -2 + i$ are drawn as illustrated below:



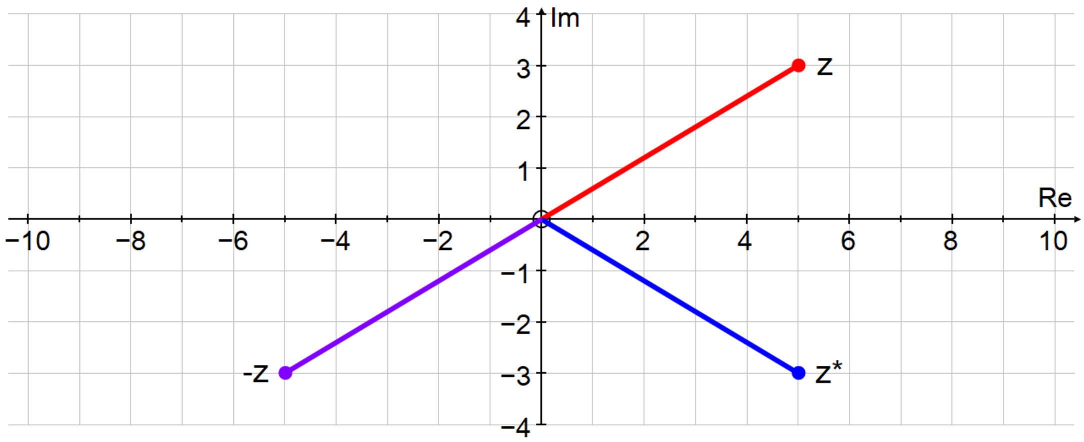
Complex numbers as points on a graph



Complex numbers as vectors

Notice that complex numbers can also be interpreted as distances from the origin (0, 0). In this case they can also be seen as vectors.

Now that we know how to plot a complex number z what can we say about the geometric effect of doing z^* and $-z$? For $z = x + iy$ we have $z^* = x - iy$ and $-z = -x - iy$. Geometrically, the effect of taking the conjugate of a complex number is simply to reflect it in the *Re* axis, and the effect of doing $-z$ is to reflect z in the *Re* axis then in the *Im* axis (or vice-versa), as illustrated in the diagram below.



Geometric effect of $z^* = x - iy$ *and* $-z = -x - iy$ *given* $z = x + iy$

Having plotted a complex number $z = x + iy$ we are now in a position to derive two new features of z: its length r , calculated via Pythagoras' theorem, and the angle θ it makes with the real axis, calculated in terms of arctan.

Hence

- length r is the modulus of the complex number and is denoted $|z|$:

$$r = |z| = \sqrt{x^2 + y^2} \tag{5}$$

- angle θ is called the argument of the complex number and is denoted arg(z):

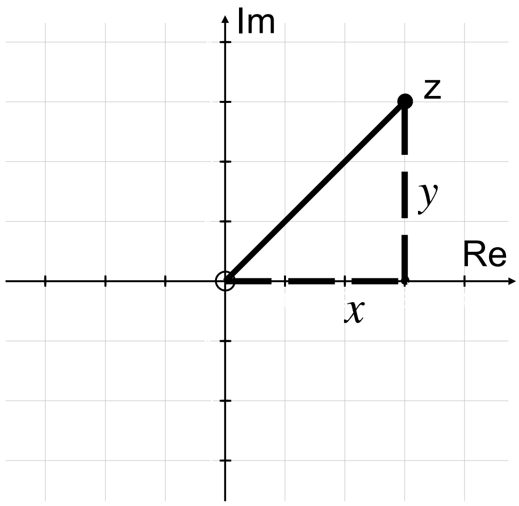
$$\theta = \arg(z) = \tan^{-1}\frac{y}{x}, \tag{6}$$

  where θ is measured in radians, and is positive if measured anticlockwise from the positive real axis, or negative is measure clockwise from the positive real axis.

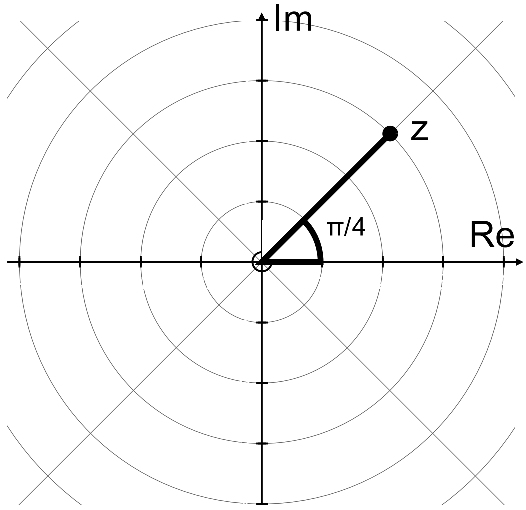
- the relationship connecting r and θ to x and y is

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta\,. \tag{7}$$

Equations (5) – (6) allow us to convert a complex number from Cartesian form to polar form, and equations (7) allow us to convert a complex number from polar form to Cartesian form. An illustration of these two forms is shown below.



Complex number z in a Cartesian form



Complex number z in a polar form

Example 1: For $z = 1 + 2i$ we have $r = |z| = \sqrt{1^2 + 2^2} = \sqrt{5}$, and $\theta = \arg(z) = \tan^{-1} 2 \approx 1.11$ radians.

Example 2: For $z = 2 - 3i$ we have $r = |z| = \sqrt{2^2 + 3^2} = \sqrt{13}$, and the angle $\theta = \arg(z) = \tan^{-1}(3/2) \approx 0.98$ radians.

Example 3: Here are some other examples of complex numbers along with their modulus and arguments

Example 4: To find the equation which satisfies $|z - 1| = 1$, for all complex numbers z we proceed as follows: let $z = x + iy$. Then $|x + iy - 1| = 1$ implies $\sqrt{(x-1)^2 + y^2} = 1$ which implies $(x-1)^2 + y^2 = 1$. This happens to be a circle of centre $(1, 0)$ and radius 1.

Example 5: To find the equation which satisfies $Re(z^2) = \left|\sqrt{3} - i\right|$ we proceed as follows: let $z = x + iy$. Then $Re(z^2) = Re(x + iy)^2 = x^2 - y^2$. Separately we have $\left|\sqrt{3} - i\right| = 2$. Hence the equation which satisfies $Re(z^2) = \left|\sqrt{3} - i\right|$ is $x^2 - y^2 = 2$ which is a hyperbola centred at $(0, 0)$.

Exercise: Find the equation which satisfies $|z| = Re(z)$.

Example 6: If $z = x + iy$ then what values of x and y satisfy $|z| - z = 2 + i$? To answer this we rewrite the equation as $\sqrt{x^2 + y^2} - (x + iy) = 2 - i$. Therefore $\sqrt{x^2 + y^2} = 2 + x + i(y - 1)$. Hence $x^2 + y^2 = [2 + x + i(y - 1)]^2 = (2 + x)^2 - (y - 1)^2 + 2i(2 + x)(y - 1)$. Comparing Re and Im parts we have

$$Re: x^2 + y^2 = (2 + x)^2 - (y - 1)^2 = x^2 - y^2 + 2x + 2y + 3.$$

Hence

$$0 = 2x + 2y + 3. \qquad (*)$$

and $Im: 0 = 2(2 + x)(y - 1)$, hence

$$0 = 4y - 2x + 2xy - 4. \qquad (**)$$

Substituting (*) into (**) and simplifying we obtain $2x^2 + 9x + 10 = 0$. This quadratic is satisfied by the values $x = -5/2$ and $x = 2$, and the respective values of y are -4 and $-7/2$.

Exercises: If $z = x + iy$, for what values of x and y is i) $|z| + 1 + 12i = 6z$ satisfied, and ii) $2z = |z| + 2i$ satisfied?

1.5 Complex roots can be located on a Cartesian graph – Part 2

The following idea is adapted and extended from the paper "Visualizing the Complex Roots of Quadratic and Cubic Equations", Alan Lipp, *The Mathematics Teacher*, Vol. 94, No. 5 (May 2001), pp. 410-413 (for further informtation see also: "Graphic Algebra", Arthur Schultze, 1909, The Macmillan Company; "Graphic Algebra (second edition)", A. Phillips, W. Beebe, 1904, Henry Holt and Company).

The real roots of any function $y = f(x)$ can be visually identified by the fact that the curve of the function crosses the x-axis. This is not the case for complex roots of $y = f(x)$. In section 1.3 we saw a way in which the complex roots of $y = f(x)$ could be identified, but these roots might be said to be hidden within the structure of the graph, rather than being explicitly visible as curves crossing axes. Therefore, it would be nice if a way could be found to visually identify comples roots in the same way as we do for real roots, and this can indeed be done as we shall now see.

1.5.1 Locating the complex roots of a quadratic equation

Let us therefore start with the quadratic $y = f(x) = x^2 + 4$. The two roots of $f(x) = 0$ are $x = \pm 2i$ and we want to be able to see these directly on Cartesian graph. In order to conduct a study of complex roots in general we need to study the equivalent complex function $w = f(z) = z^2 + 4$, where z is the complex variable $z = u + iv$.

Substituting z into $f(z)$ gives

$$w = f(z) = (u + iv)^2 + 4 = u^2 + 2iuv + i^2v^2 + 4 = 0\ ,$$

which simplifies to

$$w = f(z) = u^2 - v^2 + 4 + 2iuv = 0\ .$$

Notice that this equation is an equation in two variables u and v. If we are going to plot such an equation we need to do so in 3-dimensions. In that case the u axis will represent the real part of z, the v axis will represent values of the imaginary part of z, and w will represent the output values of {*}. Plotting the above equation directly gives us a 2D surface for all u and v. But we are not interested in this. We are interested in certain sections of the surface which relate to the real and imaginary parts of $y = f(x)$.

To obtain these sections we equate *Re* and *Im* parts left and right of the equation above to get,

$$Re(w) = u^2 - v^2 + 4 = 0, \qquad \{*\}$$

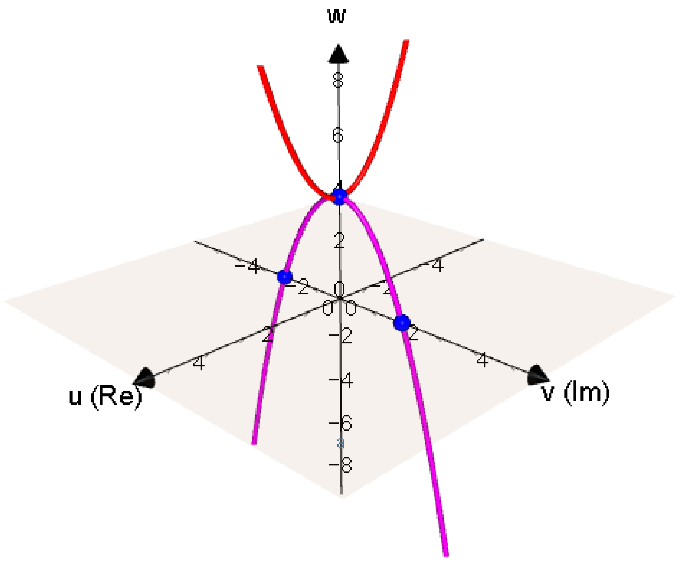
and

$$Im(w) = 2uv = 0. \qquad \{**\}$$

From {**} we have $2uv = 0$ implies $u = 0$ or $v = 0$. Now, setting $v = 0$ in {*} allows us to study the behaviour of the real part of $y = f(x)$, and setting $u = 0$ in {*} allows us to study the behaviour of the imaginary part of $y = f(x)$. Hence, by i) we have $Re(w) = Re(y) = u^2 + 4 = 0$ when $u = 0$, and $Re(w) = Im(y) = -v^2 + 4 = 0$ when $v = 0$. We can therefore plot

- $Re(w) = Re(y) = u^2 + 4$ when $v = 0$. This curve is called a *branch* of the quadratic $y = x^2 + 4$. Plotting $Re(y) = u^2 + 4$ in 3-D produces a parabolic curve in the real plane uw, as shown by the red curve in the diagram below. This function is exactly the same as $y = x^2 + 4$ in the usual Cartesian frame of reference. Since $Re(y) = u^2 + 4$ does not have any real roots we see that the red curve does not cross the uw axis, this latter plane being equivalent to the standard xy plane in 2D.
- $Re(w) = Im(y) = -v^2 + 4$ when $u = 0$. This curve is the other branch of the quadratic $y = x^2 + 4$. Plotting $Im(y) = -v^2 + 4$ in 3-D produces a parabolic curve in the plane vy, as shown by the blue curve in the diagram below. Note that $Im(y) = -v^2 + 4$ is just a plot of $Re(y) = u^2 + 4$ reflected about the vertex of this latter equation (i.e. the same type of reflection as was presented in section 1.3.1). The points of intersection of this curve with the v axis is the location of the complex roots of $y = x^2 + 4$.

The complex roots of $y = x^2 + 4$ are then located at the intersection of $Im(y) = -v^2 + 4$ and the imaginary axis v.



Plots of $Re(y) = u^2 + 4$ (in red) and $Im(y) = -v^2 + 4$ (in purple) illustrating the real and imaginary branches of $y = x^2 + 4$, along with its complex roots.

As another example let us graphically locate the complex roots of $y = x^2 - 6x + 13 = 0$. Recasting this equation in complex variable form we want to find the roots of $w = f(z) = z^2 - 6z + 13 = 0$. Again, for any complex root $z = u + iv$ we have

$$w = f(z) \quad = \quad (u + iv)^2 - 6(u + iv) + 13 = 0\,,$$
$$= \quad u^2 + 2iuv + i^2v^2 - 6u - 6iv + 13 = 0\,,$$

which, on grouping real and imaginary parts, gives

$$w = u^2 - v^2 + 6u + 13 + i(2uv - 6v) = 0\,.$$

Equating *Re* and *Im* parts left and right of this equation we obtain,

$$Re(w) = u^2 - v^2 + 6u + 13 = 0\,, \qquad (*)$$

and

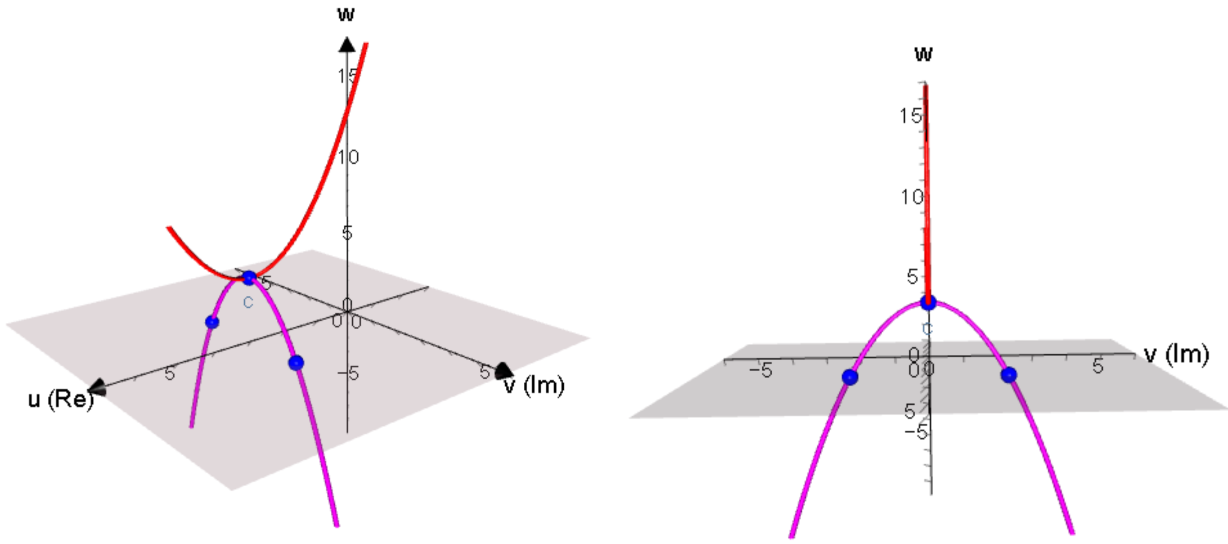
$$Im(w) = 2uv - 6v = 0 \text{ implying } u = 3 \text{ or } v = 0. \qquad (**)$$

Hence

- for $u = 3$ equation (*) gives us the branch $Re(w) = Im(y) = -v^2 + 4$. Plotting this allows us to visualise the behaviour of the imaginary part of $y = f(x)$, as can be seen in the diagram below by the purple parabolic curve parallel to the plane vw at $u = 3$;

- for $v = 0$, equation (*) gives us the branch $Re(w) = Re(y) = u^2 - 6u + 13$. Plotting this allows us to visualise the behaviour of the real part of $y = f(x)$, as can be seen in the diagram below by the red parabolic curve in the uw plane. This function is exactly the same as $y = x^2 - 6x + 13$ in the usual Cartesian frame of reference. Since $Re(y)$ happens to be function y, the former function does not have any real roots, as seen by the fact that the red curve does not cross the uw axis (this latter plane being equivalent to the standard xy plane in 2D). Also note that this equation can be rewritten as $Re(y) = (u-3)^2 + 4$, i.e. it is of the form $Re(y) = U^2 + 4$ which is just a reflection of $Im(y) = -v^2 + 4$ about the vertex of this latter equation.

The complex roots of $y = x^2 - 6x + 13$ are then located at the intersection of $Im(y) = -v^2 + 4$ and the complex plane uv, 3 units along the u axis and ± 2 units in the v axis direction, either side of $u = 3$.



Plots of $Re(y) = u^2 - 6u + 13$ (in red) and $Im(y) = -v^2 + 4$ (in purple) illustrating the real and imaginary branches of $y = x^2 - 6x + 13$, along with its complex roots.

As a final example involving quadratics let us locate graphically the complex roots of $y = (x+2)^2$, this being a repeated root: Again, $z = u + iv$ therefore we have

$$w = f(z) = (u + iv + 2)^2 = u^2 - v^2 + 4u + 4 + 2i(uv + 2v) = 0\,.$$

Equating *Re* and *Im* parts left and right of this equation we obtain,

$$Re(w) = u^2 - v^2 + 4u + 4 = 0 \qquad [*]$$

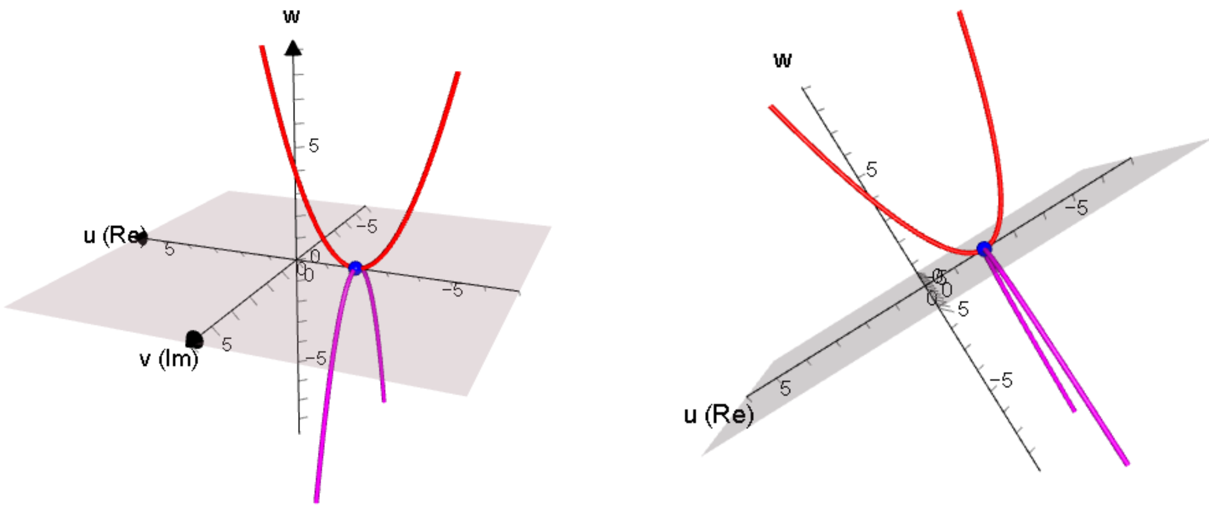
and

$$Im(w) = uv + 2v = 0 \text{ implying } u = -2 \text{ or } v = 0. \qquad [**]$$

Therefore

- for $u = -2$ equation [*] gives us the branch $Re(w) = Im(y) = -v^2$. Plotting this in 3-D produces a parabolic curve *touching* the u axis *at* $u = -2$, as shown by the purple curve in the diagram below;
- for $v = 0$, equation [*] gives us the branch $Re(w) = Re(y) = u^2 + 4u + 4$. Plotting this in 3-D produces a parabolic curve in the real plane uy, touching the u axis at $u = -2$, as shown by the red curve in the diagram below.

In this case notice that the vertices of both branches touch the real axis at $u = -2$. The real axis can be said to be tangent to both curves at $u = -2$.



Plots of $Re(y) = u^2 + 4u + 4$ (in red) and $Im(y) = -v^2$ (in purple) illustrating the real and imaginary branches of $y = x^2 + 4x + 4$, along with its complex roots.

So what we are saying is that the roots of $y = (x + 2)^2$ are $x = -2, -2$, and the roots of $w = f(z) = (z + 2)^2$ are $z = -2, -2$, i.e. the complex root z is purely real (no imaginary part), and the graphs above illustrate the visual effect of this specific situation.

The previous two examples illustrate a general feature of the graph of $y = f(x)$ when this has complex roots:

- if a quadratic has two distinct complex roots (namely, complex conjugates)
  - the real branch does not cross the u (real) axis;

- the complex branch will cross the complex uv plane at two distinct locations, these being the complex roots of $y = f(x)$, with part of the complex branch being above the complex plane, and part of it being below the complex plane;

and

- if a quadratic has a repeated complex root
    - the real branch will be tangent to the u (real) axis, this location being the root of $y = f(x)$;
    - the complex branch will be tangent to the u (real) axis, this location being the root of $y = f(x)$;

What these examples show us is that when $y = f(x)$ has complex roots, function y is composed of two curves, called *branches*, one real branch $Re(w)$ and one imaginary branch $Im(w)$. These two branches can be plotted separately, with the imaginary branch giving us the location of the complex roots of $y = f(x)$ when this branch crosses the complex plane uv (i.e. the plane commonly known as the Argand diagram).

It is left as an exercise to answer the question, What will be the location of real and complex branches w.r.t. the complex uv plane if $y = f(x)$ has real distinct roots?

1.5.2 Generalising the analysis of the complex roots of a real-valued quadratic

For the real-valued quadratic function $y = f(x) = ax^2 + bx + c = 0$, i.e. a quadratic which produces real values of y (not complex values) we transform $y = f(x)$ into its equivalent complex valued form $w = f(z) = az^2 + bz + c = 0$. We now set $z = u + iv$ giving us

$$\begin{aligned} w = f(z) &= a(u + iv)^2 + b(u + iv) + c = 0\,, \\ &= au^2 - av^2 + bu + c + 2i(auv + bv) = 0\,. \end{aligned}$$

Equating *Re* and *Im* parts left and right of this last equation we obtain,

$$Re(w) = u^2 - av^2 + bu + c = 0\,,$$

and $\qquad Im(w) = auv + bv = 0$ implying $u = -b/a$ or $v = 0$.

Therefore

- for $u = -b/a$ we have the imaginary branch $Im(y) = -av^2 + c$. Plotting this function produces a parabolic curve crossing the complex plane uv or touching the u axis depending upon whether $y = f(x)$ has complex roots or repeated roots.

The intersection of $Im(y) = -av^2 + c$ with the complex plane or the u axis gives th elocation of the complex roots of $y = f(x)$.

- for $v = 0$ we have the real branch $Re(y) = au^2 + bu + c$. Plotting this function produces a parabolic curve in the real plane uw, either lying above the u axis or touching the u axis depending upon whether $y = f(x)$ has complex roots or repeated roots. In the former case $y = f(x)$ has no real roots, and in the latter case $y = f(x)$ has repeated roots.

1.5.3 Locating the complex roots of a cubic equation

We can perform the same analysis as above to graphically locate the complex roots of cubic equations.

Example 1: As our first example let us locate the roots of $y = f(x) = x^3 - 1 = 0$. Recasting this equation in complex variable form we want to find the roots of $w = f(z) = z^3 - 1 = 0$. For any complex root $z = u + iv$ we have

$$\begin{aligned} w = f(z) &= (u+iv)^3 - 1 = 0\,, \\ &= u^3 + 3iu^2v - 3uv^2 - iv^3 - 1 = 0\,, \end{aligned}$$

which, on grouping real and imaginary parts, gives

$$w = u^3 - 3uv^2 - 1 + i(3u^2v - v^3) = 0\,.$$

Equating *Re* and *Im* parts left and right of this last equation we obtain,

$$Re(w) = u^3 - 3uv^2 - 1 = 0, \qquad ((*))$$

and

$$Im(w) = 3u^2v - v^3 = 0 \text{ implying } v = 0 \text{ or } v = \pm u\sqrt{3} \text{ implying } u = \pm v/\sqrt{3}.$$

Hence

- for $v = 0$ equation ((*)) gives us the real branch of $y = f(x)$, i.e. $Re(y) = u^3 - 1$, and is shown by the red curve in the diagrams below;

- for $u = +v/\sqrt{3}$ equation ((*)) gives us the imaginary branch of $y = f(x)$, i.e. $Im(y) = -\left(8u^3 + \sqrt{27}\right)/\sqrt{27}$, and is shown by the blue curve in the diagrams below;

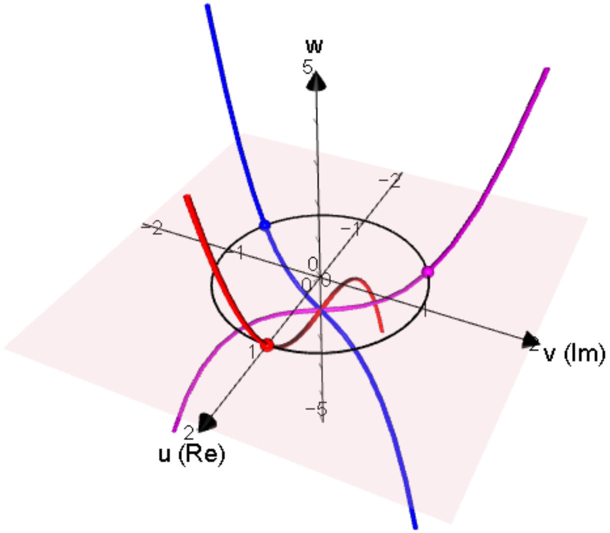
- for $u = -v/\sqrt{3}$ equation ((*)) gives us the other imaginary branch of $y = f(x)$, i.e. $Im(y) = \left(8u^3 - \sqrt{27}\right)/\sqrt{27}$, and is shown by the purple curve in the diagrams below.

Note that the two branches for $Im(y)$ can also be found from $v = \pm u\sqrt{3}$. In this case we get the simpler looking equation $Im(y) = -8u^3 - 1$. *For ease of algebra we will, from now, find the imaginary branches this way.*

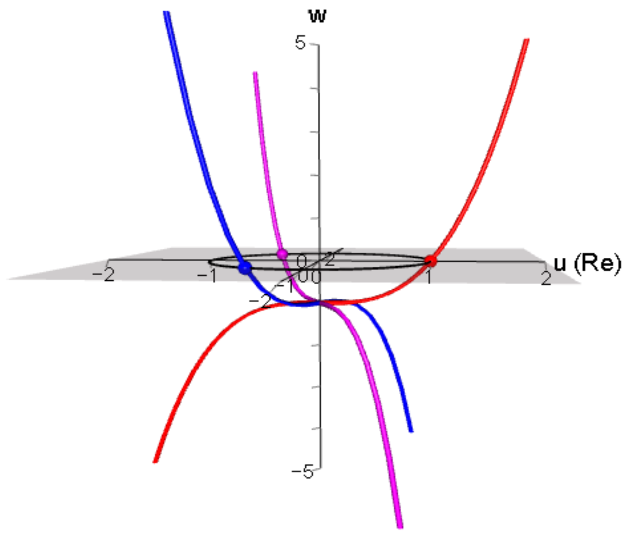
The complex roots of $x^2 — 1 = 0$ are then located at the intersection of the curves with the relevant axes or planes. So, for the red curve the intersection occurs on the Re axis at $u = 1$, shown as the red dot. This is the real root of $y = x^3 - 1 = 0$ located at $x = 1$. For the purple curve the intersection occurs on complex uv plane $(u, v) = \left(-1/2, \sqrt{3/2}\right)$, shown as the purple dot. This is the complex root of $y = x^3 - 1 = 0$ located at $x = -1/2 + i\sqrt{3}/2$.

And for the blue curve the intersection occurs on complex uv plane $(u, v) = \left(-1/2, -\sqrt{3/2}\right)$, shown as the blue dot. This is the complex root of $y = x^3 - 1 = 0$ located at $x = -1/2 - i\sqrt{3}/2$.

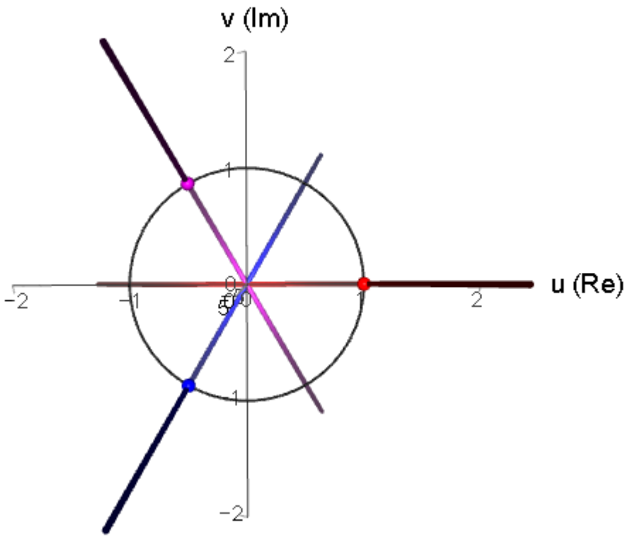
Note that all three roots lie on the circle of radius 1. This is not a coincidence, and we will cover this idea in more detail when we get to the section on roots of unity in part II of these notes.



View/perspective: 3D



View/perspective: Looking nearly onto the v plane



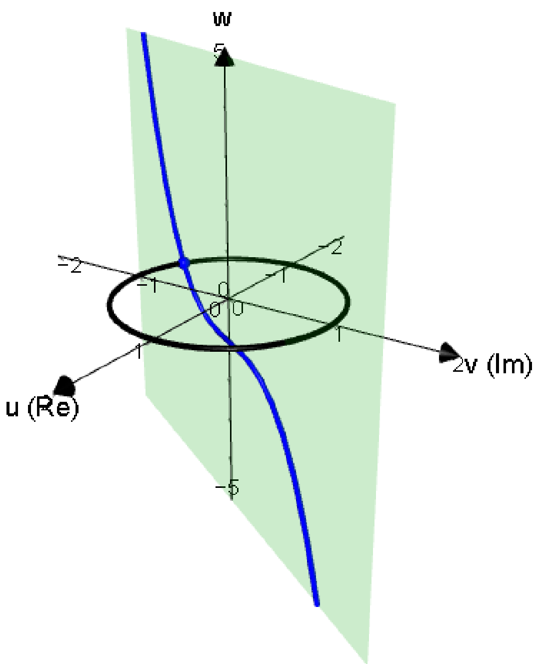
View/perspective: Looking directly down the w axis

Plots of $\boldsymbol{Re(y) = u^3 - 1}$ ***and*** $\boldsymbol{Im(y) = -8u^3 - 1}$

illustrating the real and imaginary branches of $\boldsymbol{y = x^3 - 1 = 0}$***,***

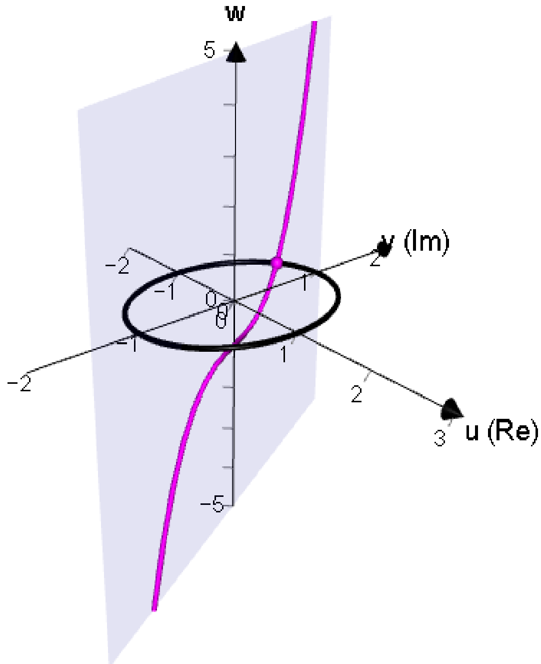
along with its complex roots.

Notice that $v = +u\sqrt{3}$ represents the plane in the direction $v = k.u$ where $k = \sqrt{3}$. Hence $Im(w)$ is plotted in this plane (shown in green below). Similarly $v = -u\sqrt{3}$ represents the plane in the direction $v = -k.u$ where $k = \sqrt{3}$. Hence $Im(w)$ is plotted in this plane (shown in blue below).



Branch $Im(y) = -8u^3 - 1$

in the plane $v = +u\sqrt{3}$



Branch $Im(y) = -8u^3 - 1$

in the plane $v = -u\sqrt{3}$

It is always the case that the imaginary parts of w will be plotted in the planes specified by the v (or equivalent u) expressions.

<u>*Example 2:*</u> As another example let us look at the cubic $y = x^3 - x = 0$. This equation has three real roots, so the question is what will the real and imaginary branches of this equation look like? As usual we recasting this equation in complex variable form as $w = f(z) = z^3 - z = 0$. For any complex root $z = u + iv$ we have

$$\begin{aligned} w = f(z) &= (u + iv)^3 - (u + iv) = 0\,, \\ &= u^3 + 3iu^2v - 3uv^2 - iv^3 - u - iv = 0\,, \end{aligned}$$

which, on grouping real and imaginary parts, gives

$$w = u^3 - 3uv^2 - u + i(3u^2v - v^3 - v) = 0\,.$$

Equating *Re* and *Im* parts left and right of this last equation we obtain,

$$Re(w) = u^3 - 3uv^2 - u = 0, \qquad ((*))$$

and

$$Im(w) = 3u^2v - v^3 - v = 0 \text{ implying } v = 0 \text{ or } v = \pm\sqrt{3u^2 - 1}.$$

Hence

- for $v = 0$ equation ((*)) gives us the real branch of $y = f(x)$, i.e. $Re(y) = u^3 - u$, and is shown by the red curve in the diagrams below;
- for $v = +\sqrt{3u^2 - 1}$ equation ((*)) gives us the imaginary branch of $y = f(x)$, i.e. $Im(y) = -8u^3 + 2u$, and is shown by the blue curve in the diagrams below;
- for $v = -\sqrt{3u^2 - 1}$ equation ((*)) gives us the other imaginary branch of $y = f(x)$, i.e. $Im(y) = -8u^3 + 2u$, and is shown by the green curve in the diagrams below.

Now, you might think that the two equations for the Im branches should be in terms of v (the imaginary component of root z) instead of being in terms of u (the real component of the complex root). So how can the two Im equations above represent imaginary branches? Well, recall the comment I made the end of the previous example for finding the roots of $x^3 - 1 = 0$, namely that we get the same results either way, so I choose to solve for the variable which is easier to solve for algebraically.

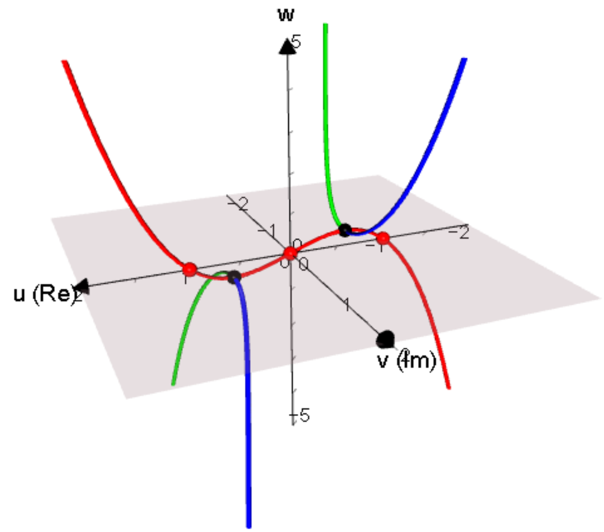
Hence the Im branches above can also be found by solving $Im(y) = 3u^2v - v^3 - v = 0$ for u. In this case the equations for "$u = \cdots$" and "$Im(y)$" will be different, but they will produce the same the complex roots and the same curves as illustrated in the graphs below. In this example it is much easier in terms of algebra to find the Im branches by solving $Im(y)$ for v. It is left as an exercise to solve $Im(y)$ for u and then find the equations of the respective branches.

Looking at the graphs below we see that neither of the imaginary branches touches or crosses the comlex plane uv. Hence $y = x^3 - x$ has no complex roots.

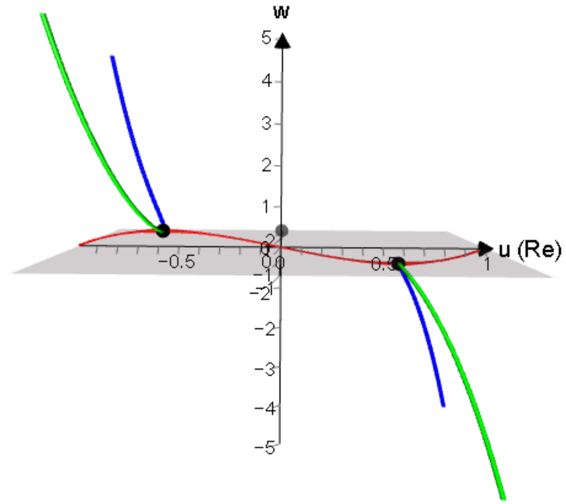
Notice that the green and blue branches are not continuous in these graphs. To see why this is so, notice that $y = \sqrt{3x^2 - 1}$ (the plane in which $Im(y)$ is graphed) is real only if $3x^2 - 1 \geq 0$, i.e. only if $x \leq -1/\sqrt{3}$ or $x \geq +1/\sqrt{3}$. This means that for all values $-1/\sqrt{3} < x < +1/\sqrt{3}$ the values of y is imaginary.

Now, remember that we are plotting curves in the 3D Cartesian system which is a region of real values only, so only those curves which have real values will be plotted here. If any curve has complex values these curves will not be seen in the graphs. So, the reason for the green curve stopping at the black dot on the negative side of the u axis, and starting again on the positive side of the u axis is because x is approaching the critical value of $\pm 1/\sqrt{3}$ beyond which no real value of y exist. The same reasons explain why the the blue curve stops at the black dot on the negative side of the u axis, and starts again on the positive side of the u axis.

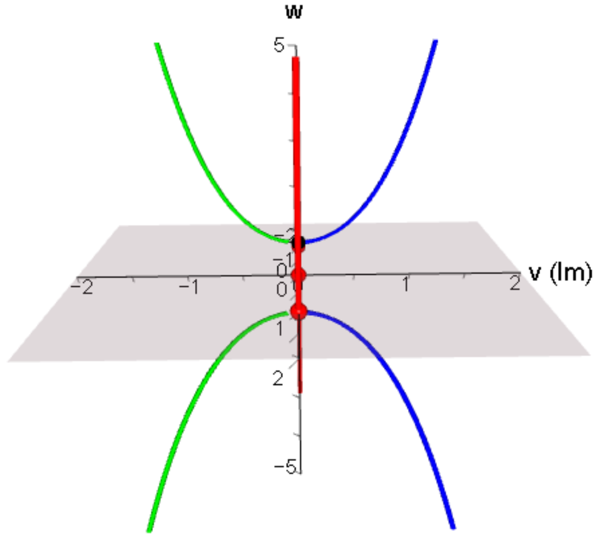
In between the stopping and starting again of the green and the blue curves lies the space of complex values, and this is where the green and blue curves will be located for value of x in $-1/\sqrt{3} < x < +1/\sqrt{3}$. If we visualise the whole of the 3D Cartesan systems as a cube, the region which the green and blue curves cannot penetrate are the whole half-cube to the right and left of the w axis respectively.



View/perspective: 3D



View/perspective: Looking nearly down the v axis



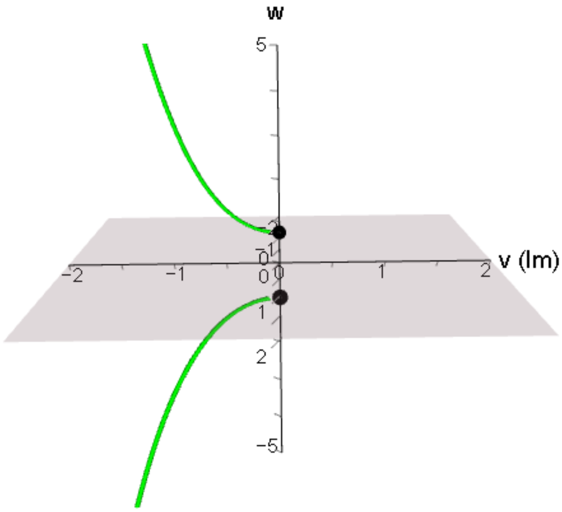
View/perspective: Looking nearly down the u axis

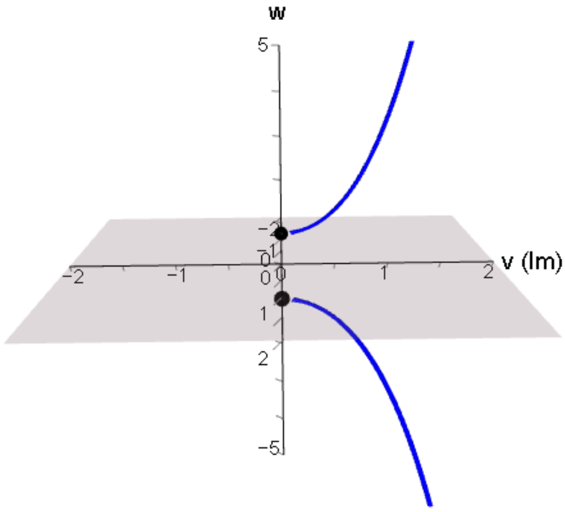
Plots of $Re(y) = u^3 - u$ and $Im(y) = -8u^3 + 2u$

illustrating the real and imaginary branches of $y = x^3 - x = 0$,

along with the lack of complex roots.

As such, these complex values cannot be represented in the Cartesian system, and is therefore seen as empty space for these two pairs of curves.





<u>*Example 3:*</u> Let us go through another example. To graphially locate the roots of $y = x^3 + x$ we proceed as usual: recasting this equation in complex variable form as $w = f(z) = z^3 + z = 0$. For any complex root $z = u + iv$ we have

$$\begin{aligned} w = f(z) \quad &= \quad (u + iv)^3 + (u + iv) = 0\,, \\ &= \quad u^3 + 3iu^2v - 3uv^2 - iv^3 + u + iv = 0\,, \end{aligned}$$

which, on grouping real and imaginary parts, gives

$$w = u^3 - 3uv^2 + u + i(3u^2v - v^3 + v) = 0\,.$$

Equating *Re* and *Im* parts left and right of this last equation we obtain,

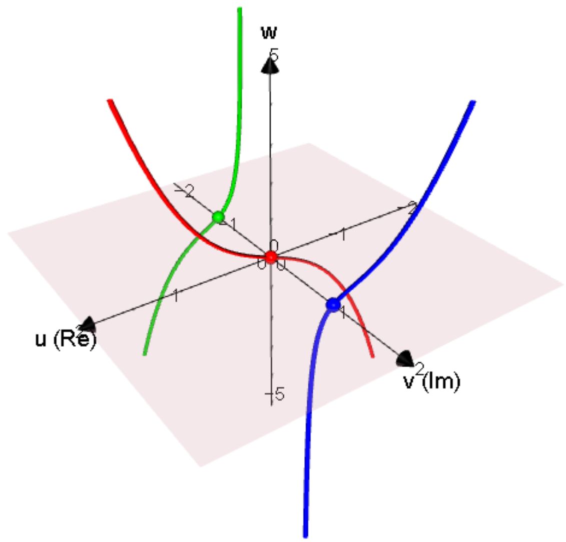
$$Re(w) = u^3 - 3uv^2 + u = 0, \qquad [[*]]$$

and

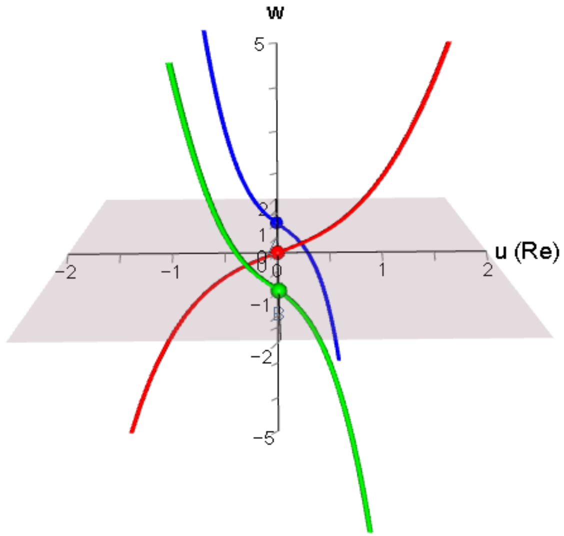
$$Im(w) = 3u^2v - v^3 + v = 0 \text{ implying } v = 0 \text{ or } v = \pm\sqrt{3u^2 + 1}.$$

Hence

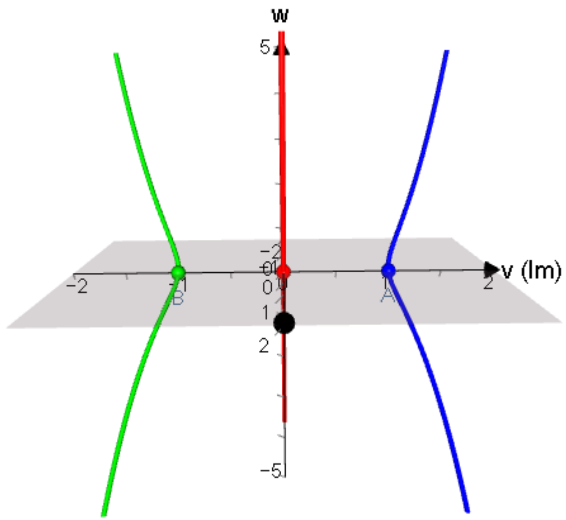
- for $v = 0$ equation [[*]] gives us the real branch of $y = f(x)$, i.e. $Re(y) = u^3 + u$, and is shown by the red curve in the diagrams below;
- for $v = +\sqrt{3u^2 + 1}$ equation [[*]] gives us the imaginary branch of $y = f(x)$, i.e. $Im(y) = -8u^3 - 2u$, and is shown by the blue curve in the diagrams below;
- for $v = -\sqrt{3u^2 + 1}$ equation [[*]] gives us the other imaginary branch of $y = f(x)$, i.e. $Im(y) = -8u^3 - 2u$, and is shown by the green curve in the diagrams below.



View/perspective: 3D



View/perspective: Looking nearly down the v axis



View/perspective: Looking nearly down the u axis

Plots of $\boldsymbol{Re(y) = u^3 + u}$ ***and*** $\boldsymbol{Im(y) = -8u^3 - 2u}$

illustrating the real and imaginary branches of $\boldsymbol{y = x^3 + x = 0}$***,***

along with the lack of complex roots.

Again, as in previous example, we could have solved $Im(w) = 3u^2v - v^3 + v = 0$ for u, and thus express $Im(y)$ in terms of v (this is left as an exercise). But again, for simplicity of lagebra, I have solved $Im(y)$ for v to get branches $Im(y)$ in terms of u.

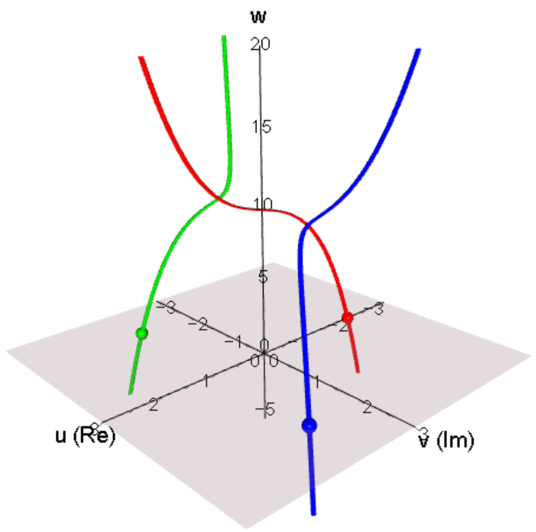
Along with being able to see the real root located at $z = 0$ (equivalent to $x = 0$ for function y) shown as the red dot in the graphs above, we can see the complex roots as being located at the intersection of the green and blue branches with the complex uv plane, i.e. $z = \pm i$, shown as green and blue dots in the graphs above.

Compared to the previous example, we see here that the green and blue branches are continuous. This is because the term $3u^2 + 1$ in $v = \pm\sqrt{3u^2 + 1}$ is always positive, so the branches based on v will always have real values.

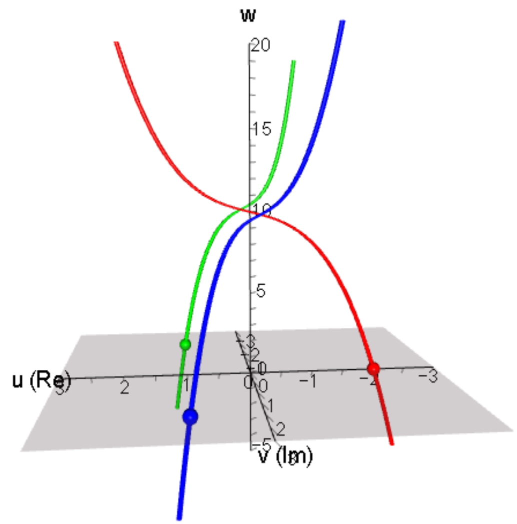
<u>*Example 4:*</u> In order to locate graphically the roots of $y = x^3 + x + 10 = 0$ the usual analysis (left as an exercise) gives use the following branches (with u and v having their usual meaning):

- for $v = 0$ we obtain the real branch of $y = f(x)$ to be $Re(y) = u^3 + u + 10$, as shown by the red curve in the diagrams below;
- for $v = +\sqrt{3u^2 + 1}$ we obtain the imaginary branch of $y = f(x)$ to be $Im(y) = -8u^3 - 2u + 10$, and is shown by the blue curve in the diagrams below;
- for $v = -\sqrt{3u^2 + 1}$ we obtain the other imaginary branch of $y = f(x)$ to be $Im(y) = -8u^3 - 2u + 10$, and is shown by the green curve in the diagrams below.

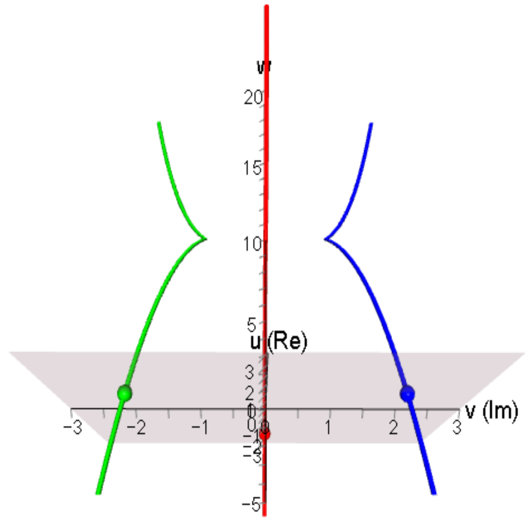
The roots can then be located visually as $z = -2$ (equivalent to $x = -2$ for function y) shown as the red dot in the graphs below, $z = 1 + 2i$ shown as the blue dot in the graphs below, and $z = 1 - 2i$ shown as the green dot in the graphs below.



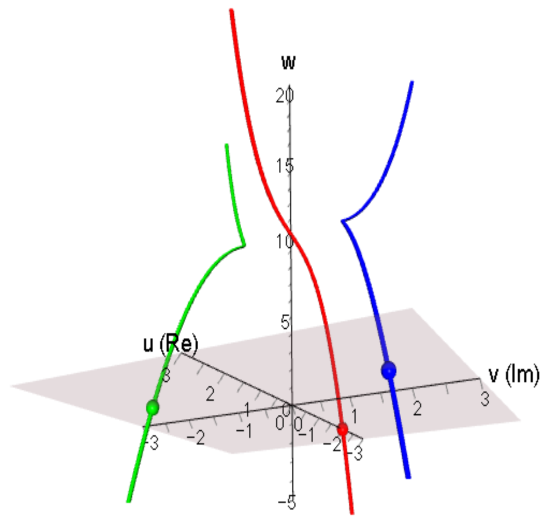
View/perspective: 3D



View/perspective: Looking nearly down the v axis



View/perspective: Looking down the u axis



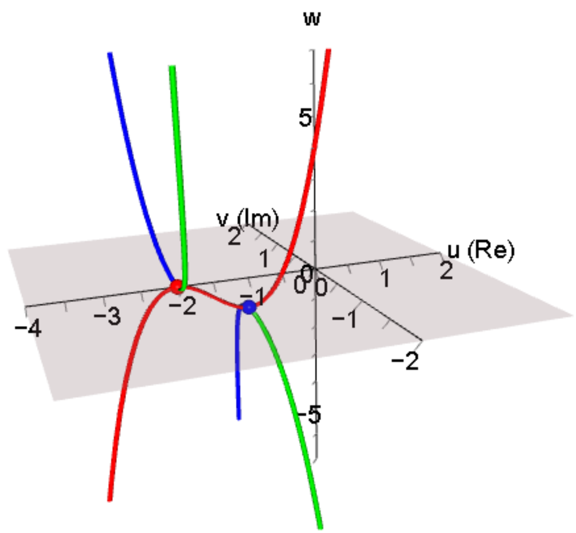
View/perspective: Looking from underneath the uv plane

Example 5: Let us now visualise the roots of $y(x) = 2x^3 + 9x^2 + 12x + 4 = 0$. This cubic has three real roots: $x = -0.5$, and a double root at $x = -2$. How will the real and imaginary branches show themselves in this case?

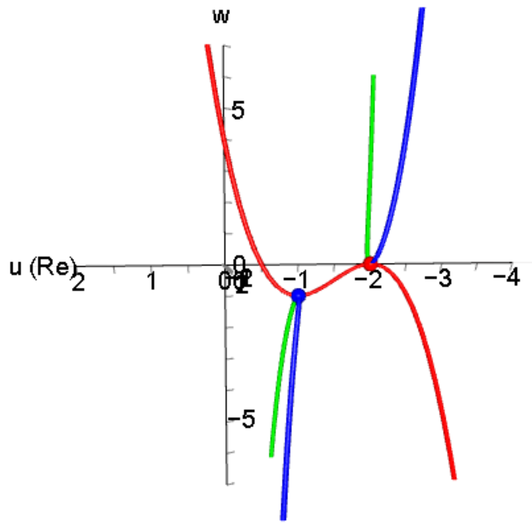
Again letting $z = u + iv$ the usual analysis (left as an exercise) gives use the following branches:

- for $v = 0$ we obtain the real branch of $y = f(x)$ to be $Re(y) = 2u^3 + 9u^2 + 12u + 4$, as shown by the red curve in the diagrams below;
- for $v = +\sqrt{3u^2 + 9u + 6}$ we obtain the imaginary branch of $y = f(x)$ to be $Im(y) = -16u^3 - 72u^2 - 105u - 50$, and is shown by the blue curve in the diagrams below;
- for $v = -\sqrt{3u^2 + 9u + 6}$ we obtain the other imaginary branch of $y = f(x)$ to be $Im(y) = -16u^3 - 72u^2 - 105u - 50$, and is shown by the green curve in the diagrams below.

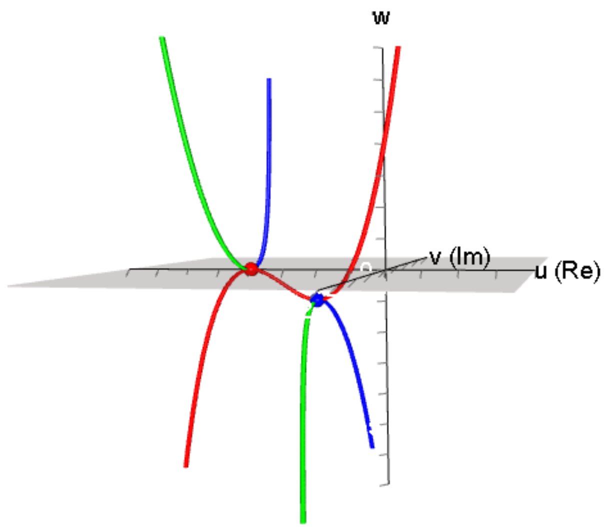
Note, as in the example of section 1.5.1 involving a quadratic with double roots, here we have the imaginary branches meeting the real branch at the double root $x = -2$. And for the remaining root of $x = -0.5$ we again see that the imaginary branches do not touch or cross the complex plane uv.



View/perspective: 3D (slightly from the negative side of uv plane)



View/perspective: Looking down the positive end of the v axis



View/perspective: Looking nearly onto the edge of the uv plane

Plots of $\boldsymbol{Re(y) = 2u^3 + 9u^2 + 12u + 4}$

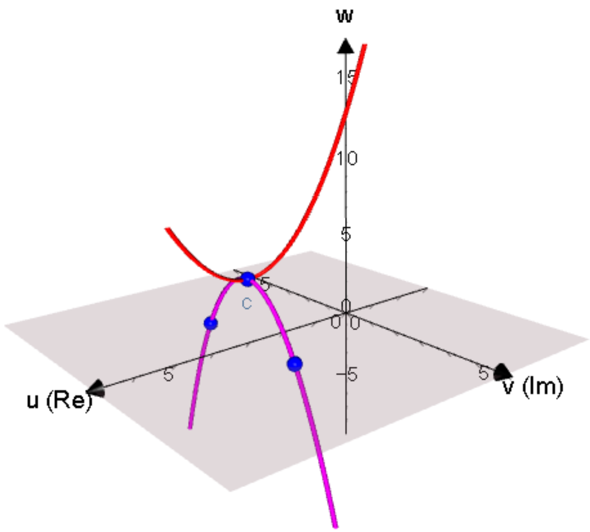
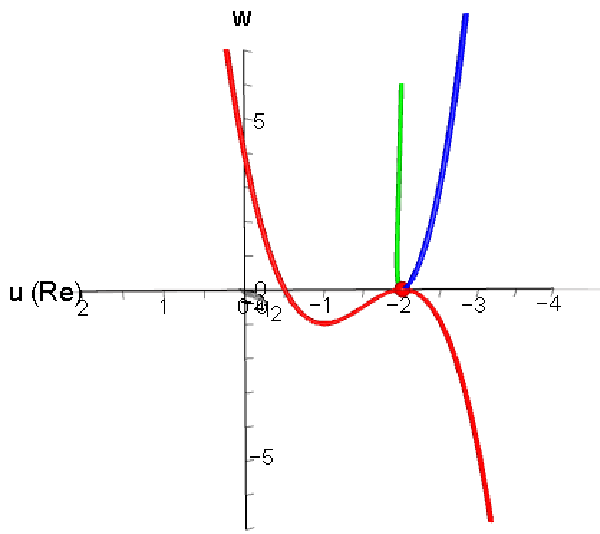
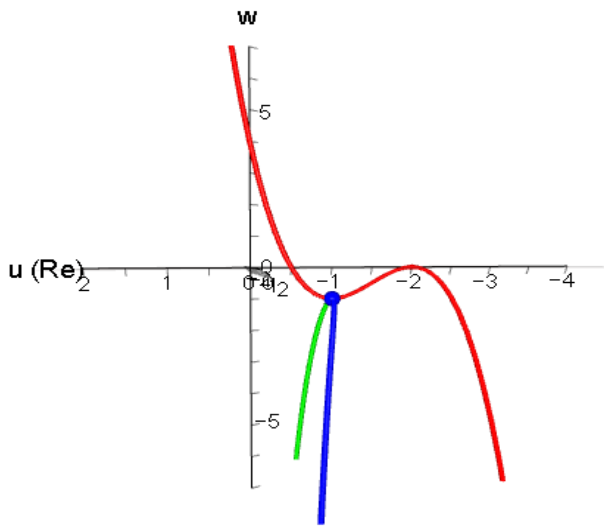
and $\boldsymbol{Im(y) = -16u^3 - 72u^2 - 105u - 50}$

illustrating the real and imaginary branches of the cubic

$$\boldsymbol{y = 2x^3 + 9x^2 + 12x + 4,}$$

along with the relevant complex roots and lack of complex roots.

One recurring theme we can see from all the work above is that of the way the real and imaginary branches behave if the roots of $y = f(x)$ are real and distinct, real and equal (double roots), or complex:

<table>
<tr><th>Complex roots only</th><th>Real double roots</th><th>Real distinct roots</th></tr>
<tr><td>If the roots of a real polynomial $y = f(x)$ are complex numbers, the real and imaginary branches of y will cross the complex plane u-v, as shown below.</td><td>If the roots of a real polynomial $y = f(x)$ are real double roots, the real and imaginary branches of y will touch the function $f(x)$, as well as touch the complex plane u-v, at the root. Specifically, the branches touch the Re axis of the complex plane, as shown below.</td><td>If the roots of a real polynomial $y = f(x)$ are distinct real roots, the real and imaginary branches of y will not touch or cross the complex plane u-v, as shown below.</td></tr>
<tr><td>
</td><td>
</td><td>
</td></tr>
<tr><td>$y = x^2 - 6x + 13, x = 3 \pm 2i$</td><td>$y = 2x^3 + 9x^2 + 12x + 4, x = -0.5, -2, -2$</td><td>$y = 2x^3 + 9x^2 + 12x + 4, x = -0.5, -2, -2$</td></tr>
</table>

1.6 On addition and subtraction of complex numbers

Having explained the rationale for needing complex numbers, and having defined what a complex number is, we now need to find out how to perform arithmetic on these numbers. Now, the thing about creating or defining a new mathematical object is that you then have to proceed systematically to explain/define how it works, how it is operated on, how it can be manipulated, etc. And given that the structure of a complex number is totally different to that of a real number (the former has two components whereas the latter has only one component) it is not automatically obvious how to perform arithmetic on complex numbers, if this can be done at all. And if it can be done, do the operations of arithmetic work in the same way on complex number as they do for real numbers?

As an example of why this needs to be done consider showing that 1 = –1. It is straightforward to do this:

$$-1 = \left(\sqrt{-1}\right)^2 \quad \Rightarrow \quad -1 = \left(\sqrt{-1}\right)\left(\sqrt{-1}\right) \quad \Rightarrow \quad -1 = \sqrt{(-1)(-1)}$$

$$\Rightarrow \quad -1 = \sqrt{1} \quad \Rightarrow \quad -1 = 1$$

But this is obviously not true, and we now meet the first of many aspects of complex numbers where the application of arithmetic is not always the same on complex numbers as it is on real numbers. This also applies even more so to functions of a complex variable $f(z)$ compared to function of a real variable $f(x)$, differentiation of $f(z)$ compared to differentiation of $f(x)$, integration of $f(z)$ compared to integration of $f(x)$, etc.

Given that the number *i* is a completely new kind of mathematical object, we have to start right from the beginning to define the operations of addition, subtraction, multiplication, and division of complex numbers (formally speaking we would even have to define what it means for two complex numbers to be equal to each other, but we will accept this as obvious).

1.6.1 Addition and subtraction of complex numbers

How do we add two real numbers? Well, we just add them, i.e. 1 + 2 = 3. Furthermore, we know that addition of real numbers is commutative, i.e. 1 + 2 = 3, and also 2 + 1 = 3. But given that a complex number is expressed as $a + ib$, the question is now, How do we go about adding two numbers of this type, and will the process of addition be commutative?

The answer to these questions is as follows: given two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ then

$$z_1 + z_2 = (a + ib) + (c + id) = (a + c) + i(b + d)\,.$$

In other words we simply add the respective real and imaginary parts. For example, $(2 + i) + (-5 - 4i) = -3 - 3i$. Then, since we know real number addition is commutative we can write

$$z_1 + z_2 = (c + a) + i(d + b) = (c + id) + (a + ib) = z_2 + z_1\,.$$

Hence addition of complex numbers is commutative. The process of subtraction of two complex numbers is defined in the same way:

$$z_1 - z_2 = (a + ib) - (c + id) = (a + c) - i(b + d)\,.$$

For example, i) $(2 + i) + (4 - 2i) = 6 - i$, ii) $-3 - (-1 + 6i) = -2 - 6i$ (notice that in this last example $b = 0$).

Theoretically we could define addition and subtraction in any way we wished to. But the question would then be as to how useful any such definition would be. This means that definitions are not (totally) arbitrary, but suffice to provide a way of manipulating new mathematical objects consistently with all previously developed mathematics.

So this way of defining addition and subtraction of two complex numbers allows us to develop a consistent mathematics, and allows us to extend the idea of addition and subtraction for real numbers.

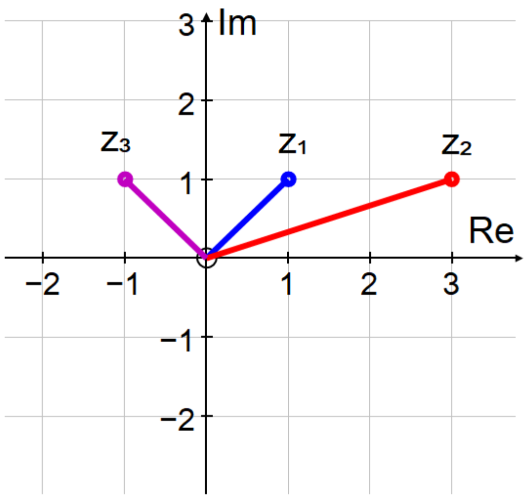
For example, given that real numbers can now be written as complex numbers where the imaginary part is 0, we can express the addition of 1 and 2 as follows:

$$\begin{aligned} 1 + 2 &= (1 + 0i) + (2 + 0i) \\ &= (1 + 2) + i(0 + 0) \\ &= 3 + 0i \\ &= 3 \end{aligned}$$

which is the same answer we get when doing 1 + 2 in $\mathbb{R}$.

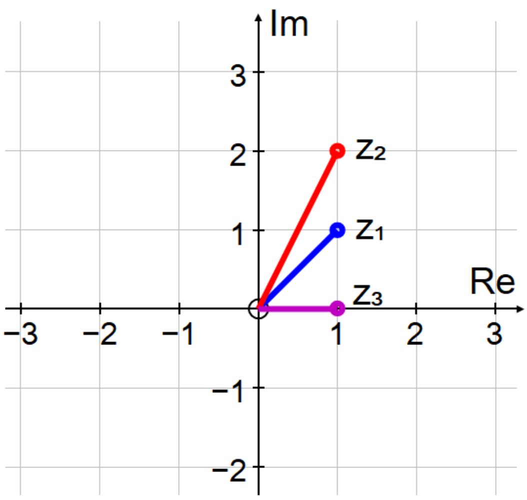
1.6.2 The geometric effect of addition/subtraction on complex numbers

Addition of complex numbers can also be intepreted geometrically. For example, let $z_1 = 1 + i$ what is the effect on z_1 of the following: $z_2 = z_1 + 2$, and $z_3 = z_1 - 2$? Well, plotting z_1, z_2, and z_3 we have the graphs below:



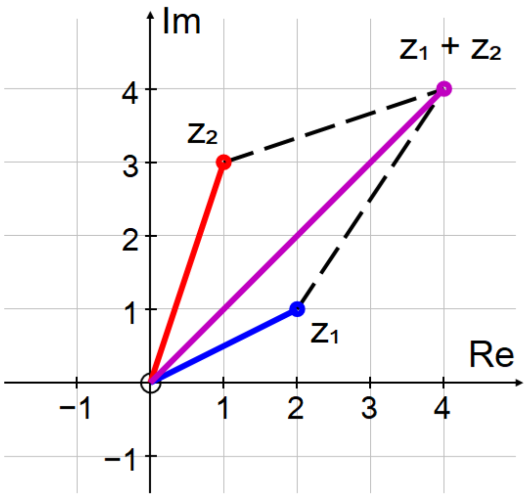
From this we see that adding 2 to z_1 translates z_1 by two unit to the right, and subtracting 2 from z_1 translates z_1 by 2 unit to the left. In general it is the case that, given $z = a + ib$, then $z \pm k$, where k is a real number, has the effect of translating/shifting z respectively to the right or left by k units.

What if we add multiples of *i* to z_1? Let us see the effect of doing $z_2 = z_1 + i$ and $z_3 = z_1 - i$:



In this case we see that the effect of adding $\pm i$ to z_1 is to translate z_1 respectively up and down by 1 unit. In general, it is the case that, given $z = a + ib$, then $z \pm ki$, where k is a real number, has the effect of translating z vertically (i.e. up or down) by k units.

Let us now study the geometric effect of adding/subtracting one complex to/from another complex number. For example, we can plot $2 + i$ and $1 + 3i$ on an Argand diagram. Plotting the sum $(2 + i) + (1 + 3i) = 3 + 4i$ shows us that addition of two complex numbers is the same as addition of vector, and can be done via the parallelogram law:

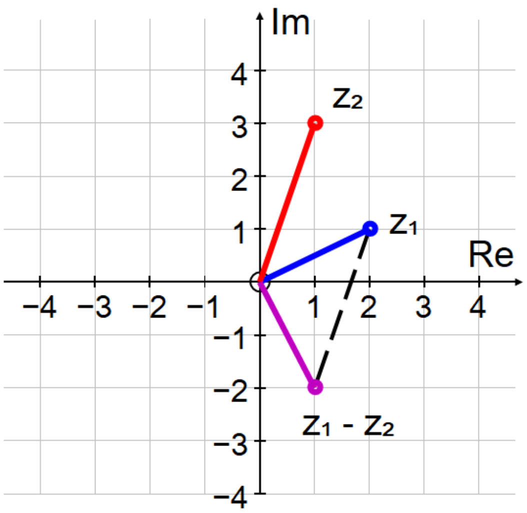


This also show us that addition of complex numbers is commutative, i.e. $z_1 + z_2 = z_2 + z_1$. It is important to know this since we cannot assume that anything about the arithmetic of real numbers transfers to the arithmetic of complex numbers.

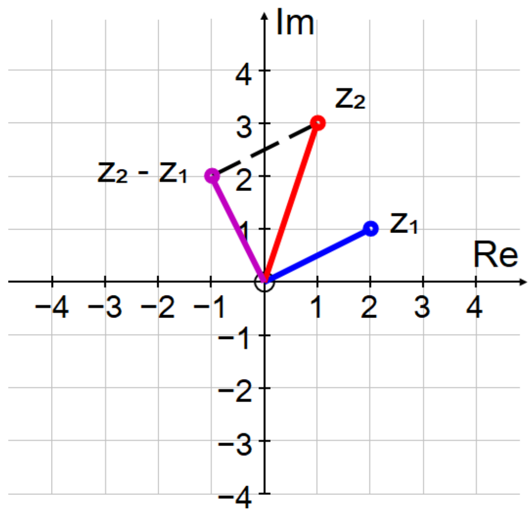
In general, given $z_1 = a + ib$ and $z_2 = c + id$ we therefore have

$$\begin{aligned} z_1 \pm z_2 &= (a + ib) \pm (c + id)\,, \\ &= (a \pm c) + i(b \pm d)\,, \\ &= (c \pm a) + i(d \pm b)\,, \\ &= (c + id) \pm (a + ib)\,, \\ &= z_2 \pm z_1\,. \end{aligned}$$

The same parallelogram law effect applies for subtraction. For example, if $z_1 = 2 + i$ and $z_2 = 1 + 3i$ then $z_2 - z_1 = -1 + 2i$ and $z_1 - z_2 = 1 - 2i$ as shown below:



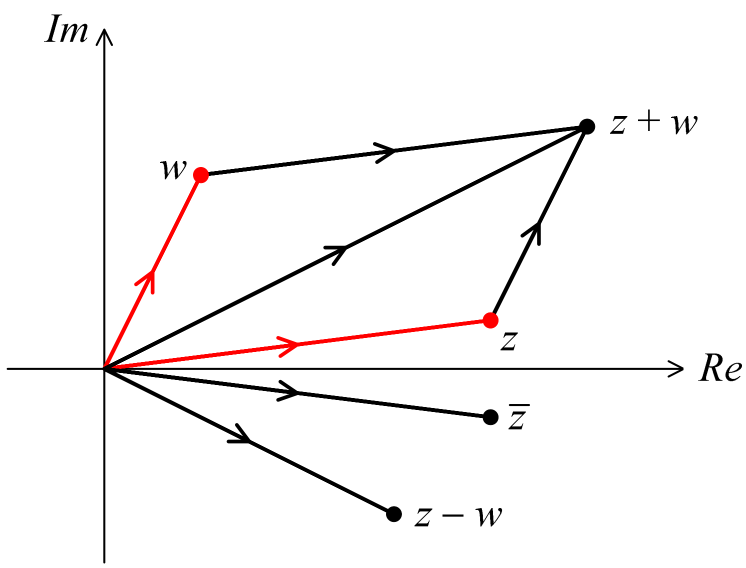
$z_1 - z_2 = 1 - 2i$



$z_2 - z_1 = -1 + 2i$

And we therefore see, both geometrically and algebraically, that subtraction is not commutative.

Below is an all-in-one diagram representing the geometric effect of the three operations of conjugation, addition, and subtraction on two complex numbers z and w.



Exercsies

1) Given $z = -2$ what is the effect of $w = z + 3i$?

2) Given $z = -1 + 2i$ and $w = 4 - i$, what is the effect of $z + w^*$ and $|w - z^*|$?

3) Given a complex number z what is the effect on z of $z + 4 - 2i$?

1.7 On Multiplication of complex numbers

1.7.1 Multiplying two complex numbers

How do we multiply two real numbers ? We just multiply them. For example 2×3 = 6. Furthermore, we know that multiplication of real numbers is commutative, i.e. 2×3 = 6, and also 3×2 = 6. But given that a complex number is expressed as $a + ib$, the question is now, How do we go about multiplying two numbers of this type, and will the process of multiplication be commutative? We also know that the real factors of 4 are 1, 2, 4. Is it possible for numbers such as $1 + i\sqrt{3}$ or $1 - i\sqrt{3}$ be factors of 4?

To answer these questions we need to define what it means to multiply two complex numbers. As such, consider two complex numbers $z_1 = a + ib$ and $z_2 = c + id$. We define the product z_1z_2 to be

$$z_1z_2 = (a + ib)(c + id) = ac + iad + ibc + i^2bd\ .$$

In other words we define multiplicaion as a simple expansion of two binomial terms. But since $i^2 = -1$ the term $i^2bd = -bd$, and we can therefore simplify the above to

$$z_1z_2 = (ac - bd) + i(ad + bc)\ .$$

Notice that the real part of z_1z_2 is not just ac but $ac - bd$, with $ad + bc$ being the imaginary part. As an example, $(2 + i)(-5 - 4i) = -10 - 8i - 5i - 4i^2 = -6 - 13i$.

Also, since we know real number addition and multiplication is commutative we can write

$$z_1z_2 = ca + icb + ida + i^2db = (c + id)(a + ib) = z_2z_1\ .$$

Hence multiplication of complex numbers is commutative. For example, $(-5 - 4i)(2 + i) = -10 - 5i - 8i - 4i^2 = -6 - 13i$, exactly as in the previous example.

Know that we know about addition and multiplication, a new question which we can ask is, Is multiplication distributitve across addion? In other words, given three complex numbers z_1, z_2, z_3 is it the case that $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$? The answer is yes, and it is left as an exercise to show this in the general case.

<u>*Examples*</u>

1) Some basic examples of multiplication of complex numbers are as follows:

- $i(5 - 7i) = 5i - 7i^2 = 7 + 5i$, since $i^2 = -1$;
- $i(4 - i) + 4i(1 + 2i) = 4i - i^2 + 4i + 8i^2 = 8i + 7i^2 = -7 + 8i$, since $i^2 = -1$;

- $(1+i)(1-2i) = 1-2i+i-2i^2 = 3-i$, since $i^2 = -1$;
- $(1+i)^2(1-i)^3 = (1+2i-i^2)(1-3i+3i^2-i^3) = 2i(-2-2i) = 4-4i$.

2) If $z_1 = 1-i$, $z_2 = -2+4i$ and $z_3 = \sqrt{3}-2i$ then we can find $Re\{2z_1^3 + 3z_2^2 - 5z_3^2\}$ as follows:

   - $2z_1^3 = 2(1-i)^3 = 2(1-i+i^2-i^3) = 2(1-i-1+i) = 0$;
   - $3z_2^2 = 3(-2+4i)^2 = 3(4-16i+16i^2) = 3(-12-16i)$;
   - $5z_3^2 = 5\left(\sqrt{3}-2i\right)^2 = 5\left(3-4i\sqrt{3}+4i^2\right) = 5\left(-1-4i\sqrt{3}\right)$.

   Therefore

$$Re\{2z_1^3 + 3z_2^2 - 5z_3^2\} = 0-36-48i+5+20i\sqrt{3} = -31+i\left(20\sqrt{3}-48\right).$$

3) If $z = x+iy$ what is $Re(z^2)$, $Im(|z|)$, $Im(\bar{z}^2 + z^2)$

   <u>*Solution:*</u> Given $z = x+iy$ then

   - $z^2 = (x+iy)^2 = x^2 + 2ixy - y^2$, therefore $Re(z^2) = x^2 - y^2$;
   - $|z| = \sqrt{x^2+y^2}$, therefore $Im(|z|) = 0$;
   - $\bar{z}^2 = x^2 - y^2 - 2ixy$, hence $\bar{z}^2 + z^2 = (x^2 - y^2 - 2ixy) + (x^2 - y^2 + 2ixy) = 2(x^2 - y^2)$. Therefore $Im(\bar{z}^2 + z^2) = 0$;
   - *Exercise:* What is $Im(\bar{z}^2 - z^2)$ and $Re(\bar{z}^2 + z^2)$?

4) Show that $Re(z_1 z_2) = Re(z_1)Re(z_2) - Im(z_1)Im(z_2)$.

   <u>*Solution*</u>

   Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then $z_1 z_2 = x_1 x_2 + i(x_1 y_2 + x_2 y_1) + i^2 y_1 y_2$. So the real part is $Re(z_1 z_2) = x_1 x_2 - y_1 y_2$.

   But $Re(z_1)Re(z_2) = x_1 x_2$ and $Im(z_1)Im(z_2) = y_1 y_2$ so $Re(z_1)Re(z_2) - Im(z_1)Im(z_2) = x_1 x_2 - y_1 y_2 = Re(z_1 z_2)$.

   <u>*Exercise:*</u> Show that $Im(z_1 z_2) = Re(z_1)Im(z_2) + Im(z_1)Re(z_2)$.

5) We have not yet studied how to take square roots of a complex number so how would we show that

$$\sqrt{1+i} = \sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{2}} + i\sqrt{-\tfrac{1}{2}+\tfrac{1}{2}\sqrt{2}}\ ?$$

Solution: Squaring both sides should show that the RHS is $1+i$. Hence

$$\begin{aligned}
\left(\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{2}} + i\sqrt{-\frac{1}{2}+\frac{1}{2}\sqrt{2}}\right)^2 &= \left(\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{2}}\right)^2 - \left(\sqrt{-\frac{1}{2}+\frac{1}{2}\sqrt{2}}\right)^2 \\
&\quad +2i\left(\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{2}}\right)\left(\sqrt{-\frac{1}{2}+\frac{1}{2}\sqrt{2}}\right), \\
&= \left(\frac{1}{2}+\frac{1}{2}\sqrt{2}\right) - \left(-\frac{1}{2}+\frac{1}{2}\sqrt{2}\right) + 2i\left(\sqrt{\frac{1}{4}}\right) \\
&= 1+i\,.
\end{aligned}$$

6) Given two complex numbers z_1 and z_2, where $z_1 \neq z_2$, we want to prove $z_1 z_2$ is a real number if and only if $z_2 = k\overline{z_1}$, for some real number k.

Solution

The term "if and only if" means that we have to prove the result in two "directions", in other words we need to prove that i) $z_1 z_2$ is a real implies $z_2 = k\overline{z_1}$ for some real number k, and ii) $z_2 = k\overline{z_1}$ for some real number k implies $z_1 z_2$ is a real.

We shall prove ii) first: Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. We use the fact that $z_2 = k\overline{z_1}$ to show that $z_1 z_2$ is real:

$$\begin{aligned}
z_1 z_2 &= z_1(k\overline{z_1})\,, \\
&= k(x_1+iy_1)(x_1-iy_1)\,, \\
&= k(x_1^2+y_1^2)\,,
\end{aligned}$$

which is indeed real.

We now prove i). In this case we need to derive $z_2 = k\overline{z_1}$ from $z_1 z_2$. Hence

$$z_1 z_2 = (x_1+iy_1)(x_2+iy_2) = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1).$$

If $z_1 z_2$ is real then $Im(z_1 z_2) = x_1 y_2 + x_2 y_1 = 0$, implying $x_2/x_1 = -y_2/y_1$. Therefore, substituting $x_2 = -x_1 y_2/y_1$ into $z_1 z_2$ we obtain

$$\begin{aligned} z_1 z_2 &= x_1\left(-\frac{x_1 y_2}{y_1}\right) - y_1 y_2 + i(0)\,, \\ &= -\frac{x_1^2 y_2 + y_1^2 y_2}{y_1}\,, \\ &= -\frac{y_2}{y_1}(x_1^2 + y_1^2)\,, \\ &= k(z_1.\overline{z_1})\,, \end{aligned}$$

where $k = -y_2/y_1$ is a real number. Hence we have $z_1 z_2 = z_1(k.\overline{z_1})$ implying $z_2 = k\overline{z_1}$. ■

Theoretically we could define multiplication in any way we wished to. But as previously mentioned the question would then be as to how useful any such definition would be. This means that definitions are not (totally) arbitrary, but suffice to provide a way of manipulating new mathematical objects consistently with all previously developed mathematics.

So this way of defining multiplication of two complex numbers allows us to develop a consistent mathematics, and allows us to extend the idea of multiplication for real numbers. Since real numbers can now be written as complex numbers where the imaginary part is 0 we have, for example, the following

$$\begin{aligned} 1 \times 2 &= (1 + 0i)(2 + 0i) \\ &= 1 \times 2 + 1 \times 0i + 2 \times 0i + 0 \times 0i^2 \\ &= 2 + 0i \\ &= 2 \end{aligned}$$

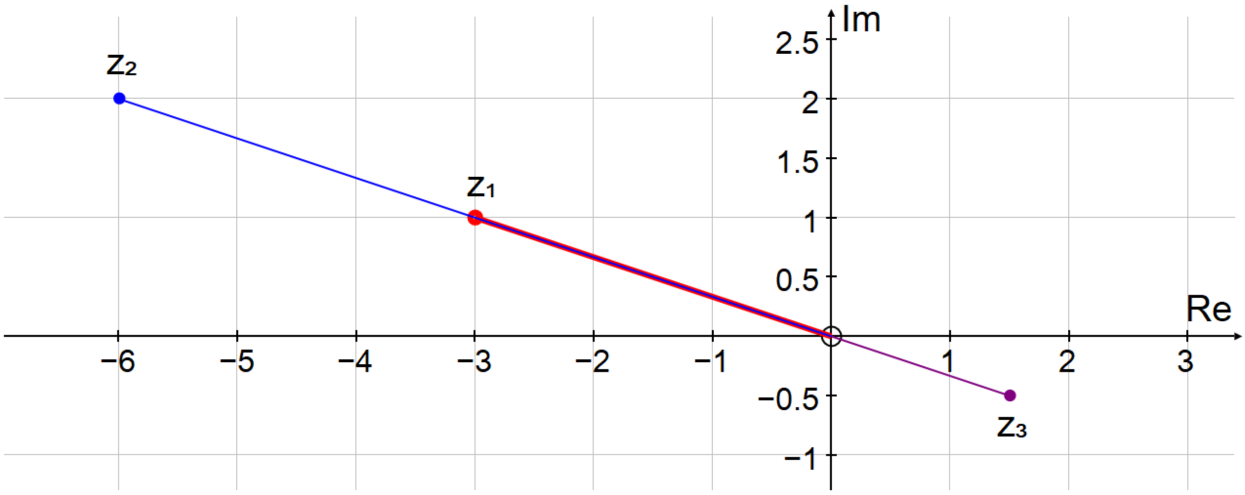
So the way of defining multiplication of complex numbers preserves the property of multiplication of real numbers.

It is important to understand why we need to define how arithmetic works on complex numbers. Since complex numbers are totally different types of numbers compared to ral numbers we cannot assume that any arithmetic on real numbers will work the same for complex numbers (go back and see what happened when we "proved" $1 = -1$ at the beginning of section 1.6).

For example, consider two real numbers x and y. Then $y^3 = x^3$ implies $y = x$. Is this also true for complex numbers? Let us test this with $\left(1 + i\sqrt{3}\right)^3$. In this case we have $\left(1 + i\sqrt{3}\right)^3 = (-2)^3$. But it is clear now that $1 + i\sqrt{3} \neq -2$. And even if, by some miracle this last equation were true (and we will see much later in the section on the exponential form of a complex number that weird and "impossible" answers such as $i^i \approx 0.20788$ are possible) we would have $1 + i\sqrt{3} = -2$ implying $i = -\sqrt{3}$ or $i^2 = 3$, which contradicts the definition $i^2 = -1$.

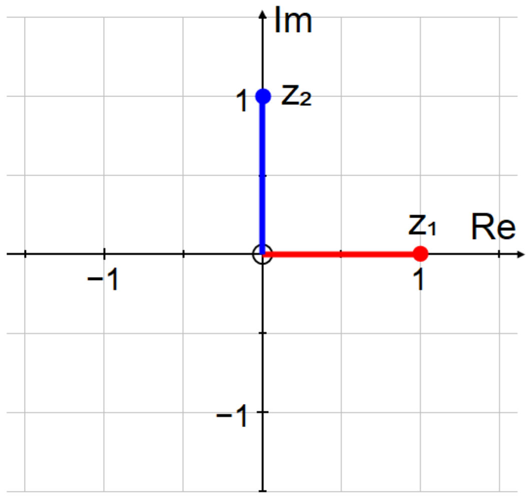
1.7.2 The geometric effect of multiplication on complex numbers

The effect of multiplication of a complex numbers by another complex number can also be interpreted geometrically. Before we get to this let us firstly consider multiplying a complex number by a real number. For example, if we have $z_1 = -3 + i$, then the effect of doing $z_2 = 2.z_1 = 2(-3 + i) = -6 + 2i$ is to stretch z_1 by a factor of 2. If we multiply z_1 by –½ we get $z_3 = -0.5z_1 = -0.5(-3 + i) = 1.5 - 0.5i$ which is a squeezing of z_1 by a factor of ½, pointing in the opposite direction to z_1.



We therefore see that the geometric effect of this is to simply stretch or squeeze our complex number. That's it. There is no more to it.

Let us now look at the effect of multiplying $z_1 = 1$ by i. Performing the multiplication gives $z_2 = i.z_1 = i$, which can be represented graphically as seen below:



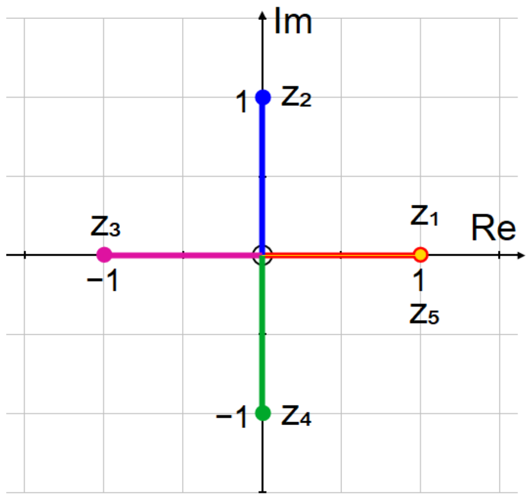
So the effect of multiplying z_1 by i is to rotate z_1 by $\pi/2$. If we repeat this by multiplying z_2 by i to get z_3, and then by multiplying z_3 by i to get z_4, and then by multiplying z_4 by i to get z_5 we have the following:

$$z_3 = i.z_2 = i.i = i^2 = -1$$

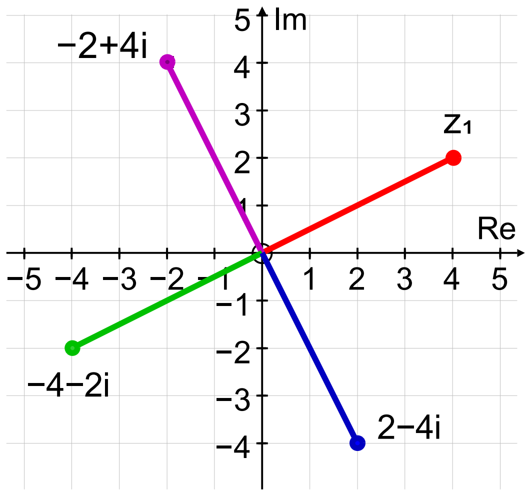
$$z_4 = i.z_3 = i(-1) = -i$$

$$z_5 = i.z_4 = i(-i) = -i^2 = 1$$

all of which can be represented graphically below:



The effect of rotating by $\pi/2$ is true generally: if we multiply any complex number z by i the number z gets rotated by $\pi/2$. For example, if $z_1 = 4 + 2i$ is multiplied by i we get $z_2 = iz_1 = i(4 + 2i) = -2 + 4i$. Multiplying again by I we get $z_3 = i.z_2 = i(-2 + 4i) = -4 - 2i$. Multiplying again by i we get $z_4 = i.z_3 = i(-4 - 2i) = 2 - 4i$, etc. As can be seen from the graph below, each time we multiply by i we rotate the given complex number z by $\pi/2$.

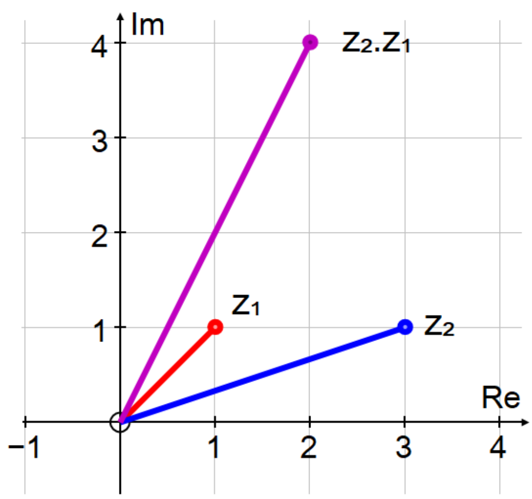


We have seen how multiplying a complex number z by i has the effect of rotating z by $\pi/2$, more specifically z is rotated anticlockwise by $\pi/2$. What then will be the effect of multiplying z by $-i$? Here the effect will be to rotate z by $\pi/2$ in a clockwise direction. Successively multiplying z by $-i$ will then have the effect of successively rotating z by $\pi/2$ in a clockwise direction.

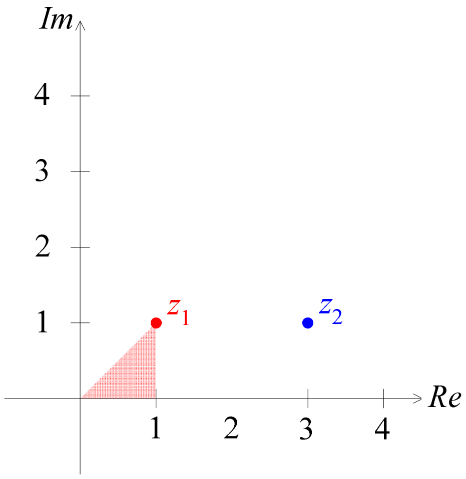
So in summary we can say that

- Multiplying any complex number z by i, i^2, i^3, i^4 has the effect of rotating z by $\pi/2$, π, $3\pi/2$, and 2π respectively, i.e. an anticlockwise rotation;
- Multiplying any complex number z by $-i$, $-i^2$, $-i^3$, $-i^4$ has the effect of rotating z by $-\pi/2$, $-\pi$, $-3\pi/2$, and -2π respectively, i.e. a clockwise rotation;

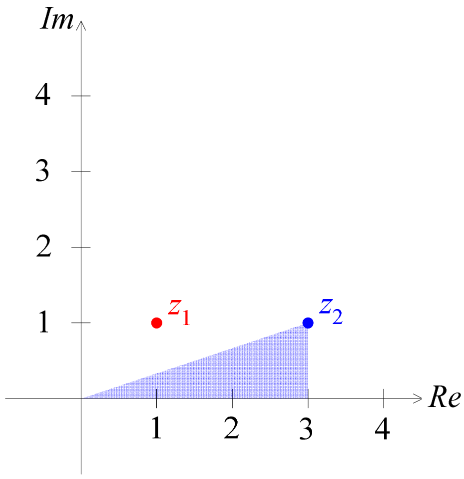
We are now in a position to describe the geometric effect of multiplying any arbitrary complex number z_1 by any other arbitrary complex number z_2. To illustrate this let us look at multiplying $z_1 = 1 + i$ by $z_2 = 3 + i$. Algebraically we obtain $z_2 z_1 = (3 + i)(1 + i) = 2 + 4i$, illustrated graphically as below:



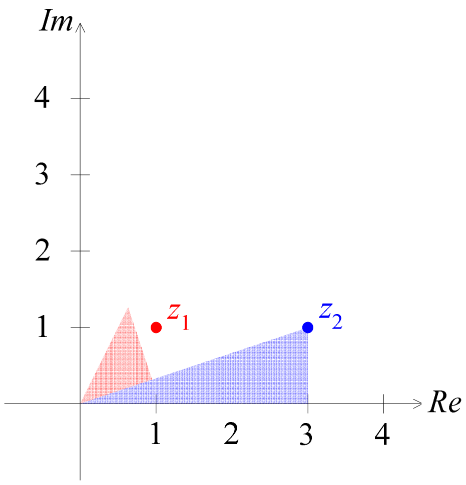
How are we going to interpret the geometric effect of this multiplication? Well, briefly we can see that multiplying z_1 by z_2 has rotated z_1 as well as stretched it. Informally, this can be explained as follows: create a right-triangle with Oz_1as hypotenuse, and a right-triangle with Oz_2 as hypotenuse. Then rotate the z_1 triangle so that its base lies on the hypotenuse of z_2. Then stretch the z_1 triangle until its base meets z_2. The vertex of this new rotated and stretched triangle is the product z_2z_1.



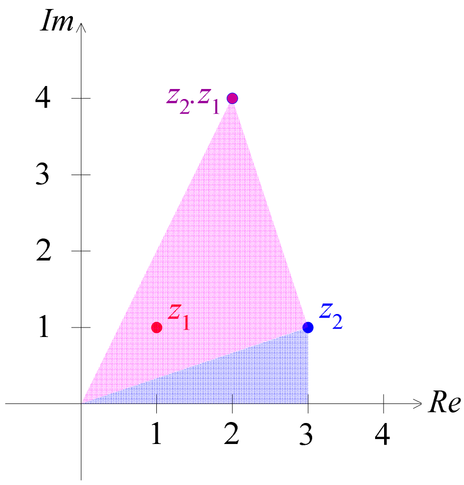
The triangle formed by z_1



The triangle formed by z_2



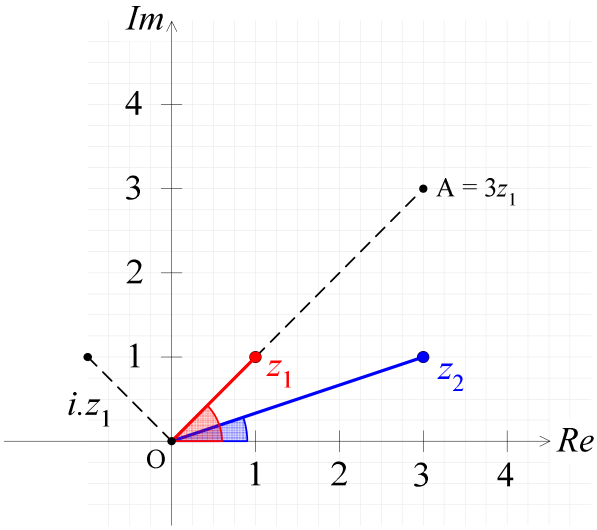
Rotating the z_1 triangle so that its base lies on the hypotenuse of z_2



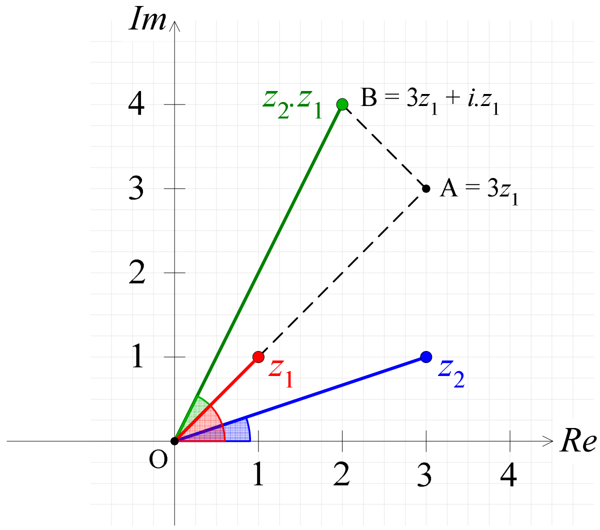
Scaling the rotated triangle z_1 so that its base meet z_2

We know need to know how/why this works. To help us in this we will refer to the diagram below, along with the fact that $z_2z_1 = (3+i)(1+i) = 3(1+i) + i(1+i)$.

What this tells us is that we scale z_1 by a factor of 3 and seperately rotate z_1 by 90° and scale this by a factor of 1. We then add (in the sense of vector addition) both of these transformations.



Scaling z_1 and scaling a rotated z_1



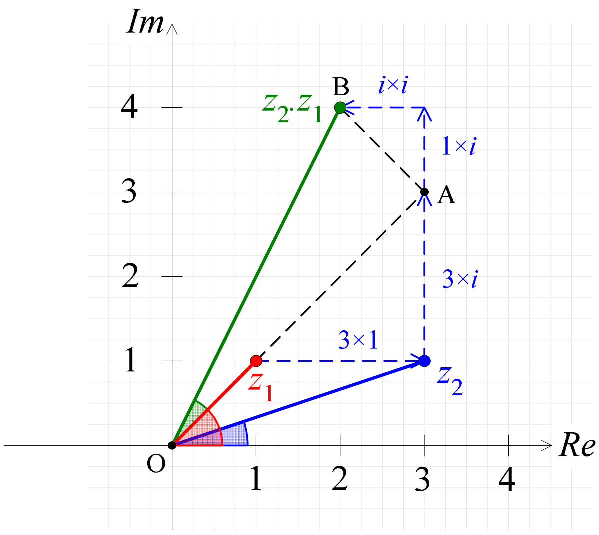
Adding the scaled and rotated components

This combination of stretching and rotating gives us $z_2 z_1$. The rotation effect is such as to add the angle z_2 makes with the *Re* axis with the angle z_1 makes with the *Re* axis. Also, since we are scaling up z_1 by the amount z_2, the length of $z_2 z_1$ is simply the product of the lengths of z_1 and z_2.

The effect of addition of angles and multiplication of lengths can be seen by the fact that triangle OAB, arrived at by the vector addition of the *Re* and *Im* components of $z_2 z_1$, viz,

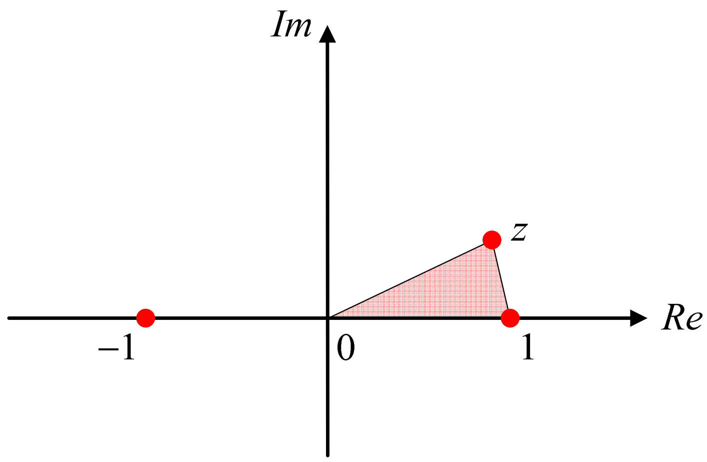
$$z_2 z_1 = (3+i)(1+i) = (3 \times 1 + 3 \times i) + (i \times 1 + i \times i)\,,$$

(illustrated below) is similar to the right-triangle formed by z_2.



1.7.3 The effect of continual multiplication by a complex number

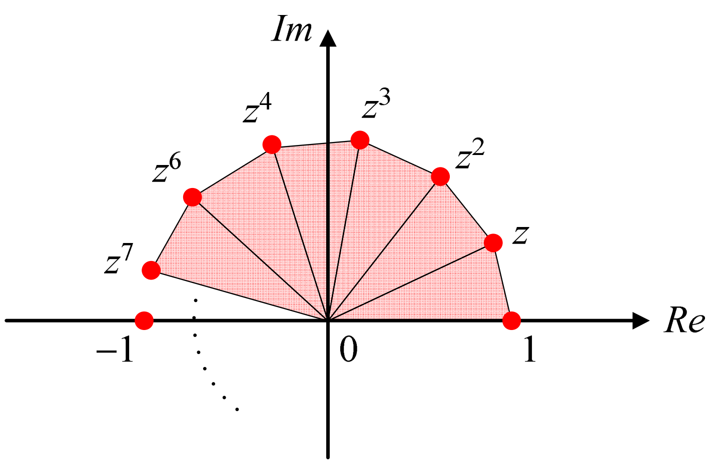
Consider the complex number such that $|z| = 1$. What is the geometric effect of performing z^n when n is a positive integer? To answer this let us locate z on a unit circle, as shown below.



Since $|z| = 1$ the line from the centre to point z has length 1. Therefore, the triangle formed from $(0, 0)$ to $(1, 0)$ to z will be isosceles, with two sides being of length 1. These two sides are effectively the base and the hypotenuse of the triangle.

Recall that multiplication of two complex numbers z_1 and z_2 involves placing the base of triangle of z_1 onto the hypotenuse of triangle z_2, and then stretching the base of z_1 to match the length of z_2. In the case of z^2 we place the base of the triangle formed by z onto the hypotenuse of the triangle formed by z. Since both of these sides are of length 1 no stretching is required. Therefore the effect of multiplying z by z, i.e. performing z^2, is simply to rotate z.

So when we take a complex number z, such that $|z| = 1$, and integer-power it up, i.e. square it, cube it, fourth power it, etc., , the result will be seen to rotate z but to not stretch it:

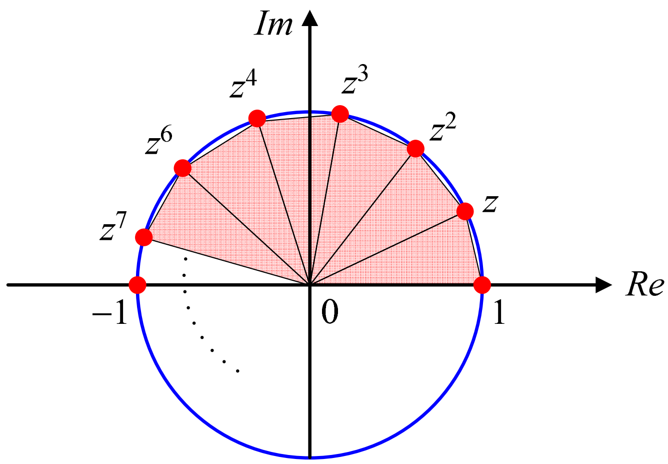


As an example consider $z = 0.8 + 0.6i$. here we have $|z| = 1$. Integer-powering z we obtain the following up to power 6:

$z^2 = 0.28 + 0.96\mathrm{i}$ $\quad$ $z^3 = -0.352 + 0.936\mathrm{i}$ $\quad$ $z^4 = -0.8432 + 0.5376\mathrm{i}$

$z^5 = -0.99712 - 0.07584\mathrm{i}$ $\quad$ $z^6 = -0.752192 - 0.658944\mathrm{i}$

etc. It seems as if the integer powers of z follow the path of the circumference of the unit circle, and this is indeed the case, and is illustrated below. This can be shown simply by finding the equation which satisfies $|z| = 1$ (this is left as an exercise).

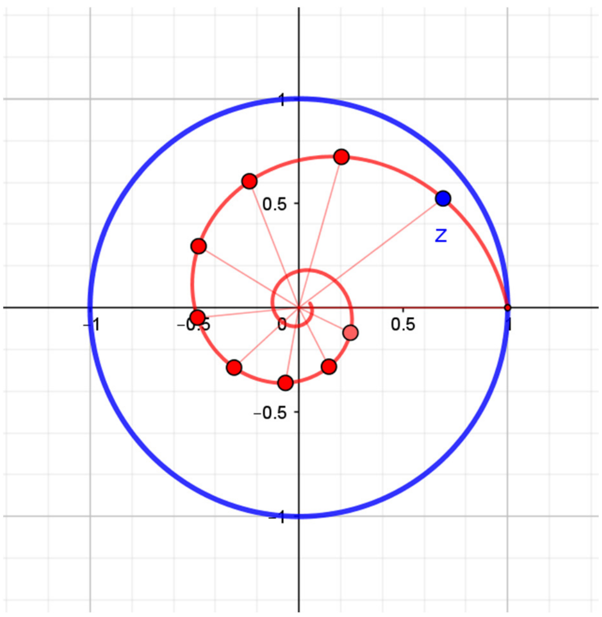


Examples such as that above will be studies in more detail in *Complex Number II* under the section titled loci. In the case of the example above the geometric effect of integer powers of z can informally be called the *power circle*.

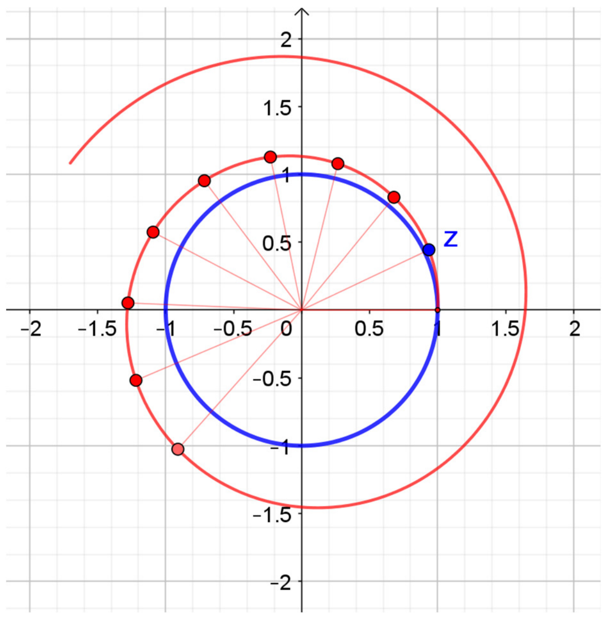
An interesting question to ask is, What path do integer powers of z follow if $|z| \gtrless 1$? Well, if $|z| > 1$ the path is a spiral lying outside the unit circle, with the spiral becoming more and more open the greater the modulus of z. If $|z| > 1$ the path is a spiral lying inside the unit circle, with the spiral becoming more and more closed the less the modulus of z.

This is illustrated below for $z = 0.69 + 0.52i$ ($|z| < 1$) and for $z = 0.94 + 0.44i$ ($|z| > 1$), where the sequence of red dots after z represent the powering of z from 2 to 8. Such a path might informally be called a *power spiral*. A summary of the modulus and arguments (in radians) of each power of z is shown in the table below:

Complex number $z = 0.69 + 0.52i$	$\lvert \boldsymbol{z^n} \rvert$	$\mathbf{arg}(\boldsymbol{z^n})$	**Complex number** $z = 0.94 + 0.44i$	$\lvert \boldsymbol{z^n} \rvert$	$\mathbf{arg}(\boldsymbol{z^n})$
z	0.864	0.646	z	1.038	0.438
z^2	0.747	1.292	z^2	1.077	0.876
z^3	0.645	1.937	z^3	1.118	1.313
z^4	0.557	2.58	z^4	1.16	1.751
z^5	0.482	3.23	z^5	1.204	2.189
z^6	0.416	3.87	z^6	1.25	2.627
Each higher power is 0.646 radians further along anticlockwise, and closer to 0 in length.			Each higher power is 0.438 radians further along anticlockwise, and further away from 0 in length.		



A power spiral for z^n where $|z| < 1$



A power spiral for z^n where $|z| > 1$

1.7.4 Finding the square root of a complex number

We now have enough development in complex numbers to find the square root of a complex number.

Example 1

Suppose we have $z = 5 + 12i$ and we now want to find $\sqrt{z}$. We can do this as follows:

$$\sqrt{5 + 12i} = a + ib\,.$$

Squaring both sides gives

$$\begin{aligned} 5 + 12i &= (a + ib)^2\,, \\ &= (a^2 - b^2) + i(2ab)\,. \end{aligned}$$

We now compare *Re* and *Im* coefficients to get

$$5 = a^2 - b^2\,, \tag{8}$$

$$\text{and} \qquad 12 = 2ab \qquad \text{implying} \qquad 6 = ab\,. \tag{9}$$

Substituting (9) into (8) we obtain

$$5 = a^2 - \left(\frac{6}{a}\right)^2\,,$$

which simplifes to

$$a^4 - 5a^2 - 36 = 0\,. \tag{10}$$

Using the quadratic formula on (10) we get

$$a^2 = \frac{5 \pm \sqrt{25 + 4 \times 36}}{2} = \frac{5 \pm \sqrt{169}}{2}\,.$$

However, $5 - \sqrt{169} < 0$, i.e. a^2 is negative, leading to a being complex. But a is a real number so the negative square root case is not valid. Therefore

$$a^2 = \frac{5 + \sqrt{169}}{2} = 9\,,$$

implying

$$a = \pm 3\,, \tag{11}$$

$$b = \pm 2\,. \tag{12}$$

But by (9) we know that $a.b$ is positive, so a and b must have the same sign, either $a, b > 0$ or $a, b < 0$. Therefore $a = 3$ and $b = 2$, or $a = -3$ or $b = -2$, and the two roots of $5 + 12i$ are $3 + 2i$ and $-3 - 2i$.

Suppose, on the other hand that we want to find the square root of $5-12i$. The majority of the steps in solving this are the same as above except that (9) now becomes

$$-6 = ab \quad \text{implying} \quad -\frac{6}{a} = b \tag{13}$$

This again leads to $a^4 - 5a^2 - 36 = 0$, and ultimately to $a = \pm 3$ and $b = \pm 2$. However, by the left hand expression in (1311) we know that *a.b* is negative, so *a* and *b* must have opposite signs, either $a > 0$ and $b < 0$, or vice-versa Therefore $a = -3$ and $b = 2$, or $a = 3$ or $b = -2$, and the two roots of $5 - 12i$ are $-3 + 2i$ and $3 - 2i$.

Example 2

Finding the square root of $z = 9 + 4i$ we have:

$$\sqrt{9+4i} = a + ib\,.$$

Squaring both sides gives

$$\begin{aligned} 9 + 4i &= (a+ib)^2\,, \\ &= (a^2 - b^2) + i(2ab)\,. \end{aligned}$$

We now compare *Re* and *Im* coefficients to get

$$9 = a^2 - b^2\,, \tag{14}$$

$$\text{and} \qquad 4 = 2ab \qquad \text{implying} \qquad 2 = ab\,. \tag{15}$$

We could now substitute (15) into (14) for *b*, but let us do something else instead. Let us create a third equation, which will end up making our subsequent algebra easier. This third equation is based on the modulus of z. In other words, since $\sqrt{9+4i} = a + ib$, we have $9 + 4i = (a+ib)^2 = (a^2 - b^2) + i(2ab)$. Taking the modulus of both sides gives

$$\sqrt{9^2 + 4^2} = \sqrt{(a^2 - b^2)^2 + (2ab)^2} \ \text{implying} \sqrt{97} = a^2 + b^2 \tag{16}$$

We can now add (14) and (16) and take the square root to obtain

$$a = \pm\sqrt{\frac{1}{2}\left(9 + \sqrt{97}\right)}\,,$$

whence we can find *b* from to be

$$b = \pm 2\sqrt{\frac{2}{9+\sqrt{97}}}\,.$$

Again note that by (15) the signs of a and b have to be the same, so we have the roots of $9+4i$ to be

$$z_1=\sqrt{\frac{1}{2}\left(9+\sqrt{97}\right)}+2i\sqrt{\frac{2}{9+\sqrt{97}}} \qquad \text{and} \qquad z_2=-\sqrt{\frac{1}{2}\left(9+\sqrt{97}\right)}-2i\sqrt{\frac{2}{9+\sqrt{97}}}\ .$$

We can generalise the above examples as follows: the number $\sqrt{x+iy}$ is a complex number $a+ib$ such that $x+iy=(a+ib)^2$. Hence

$$\begin{aligned} x+iy &= (a+ib)^2\,, \\ &= (a^2-b^2)+i(2ab)\,. \end{aligned}$$

Comparing *Re* and *Im* coefficients we get

$$x=a^2-b^2\,, \tag{17}$$

$$\text{and} \qquad y=2ab \qquad \text{implying} \qquad \frac{y}{2a}=b\,, \tag{18}$$

from which we obtain

$$x=a^2-\left(\frac{y}{2a}\right)^2,$$

which simplifes to

$$4a^4-4xa^2-y^2=0\,. \tag{19}$$

Using the quadratic formula on we obtain

$$a^2=\frac{4x\pm\sqrt{16x^2+16y^2}}{8}=\frac{x\pm\sqrt{x^2+y^2}}{2}\,.$$

However, $\sqrt{x^2+y^2}\geq x$, so in general $x-\sqrt{x^2+y^2}$ will be negative, i.e. a^2 is negative leading to a being complex. But a is a real number so the negative square root case is not valid. Therefore

$$a^2=\frac{x+\sqrt{x^2+y^2}}{2}\,,$$

Substituting this back into $y=2ab$ we obtain

$$b^2=\frac{-x+\sqrt{x^2+y^2}}{2}\,,$$

hence

$$a = \pm\sqrt{\frac{x + \sqrt{x^2 + y^2}}{2}}, \qquad\qquad b = \pm\sqrt{\frac{-x + \sqrt{x^2 + y^2}}{2}}.$$

But we know that if $y = 2ab$ is positive then a and b must have the same sign, and if $y = 2ab$ is negative then a and b must have opposite signs. Therefore,

$$b \geq 0 \quad \text{implies} \quad \sqrt{x + iy} = \pm\left(\sqrt{\frac{x + \sqrt{x^2 + y^2}}{2}} + i\sqrt{\frac{-x + \sqrt{x^2 + y^2}}{2}}\right),$$

$$b < 0 \quad \text{implies} \quad \sqrt{x + iy} = \pm\left(\sqrt{\frac{x + \sqrt{x^2 + y^2}}{2}} - i\sqrt{\frac{-x + \sqrt{x^2 + y^2}}{2}}\right).$$

1.8 On division of complex numbers

1.8.1 The division of two complex numbers

How do we divide two real numbers ? We just divide them. For example 7÷2 = 3.5. Let us look more closely at the fraction 7/2. We know that one way to interpret this fraction is by saying that we are looking to find how many 2s go into 7. And by repeated subtraction we find that there are three 2s, with 1 left over. So

$$\frac{7}{2} = 3\frac{1}{2}.$$

By rewriting ½ as 10×0.1/2 we can continue this process and ask how many 2s go into 10 tenths, from which we find there to be 5 tenths. Hence, we then end up with

$$\frac{7}{2} = 3.5$$

All of this is basically a process of repeated subtraction.

But given that a complex number is expressed as $a + ib$, the question is now, How do we go about dividing two numbers of this type? For example, what is the answer to $(2 + i) \div (3 + 2i)$? What does it mean to ask, How many "$3 + 2i$"s go into $2 + i$? Well, it doesn't mean anything. The concept of "how many … go into …?" is really a concept of magnitude (applying only to real numbers), and complex numbers are not magnitudes but vector-like objects. A real number is a magnitude type of object whereas a complex number is a *magnitude-and-direction* type of object. So how are we going to divide a number which has magnitude and direction?

To answer this let us return to our real number fraction $7/2$, and consider an alternative way to performing the division. If this division is possible then it will be some answer m. This means we can write

$$\frac{7}{2} = m\,.$$

Hence we have

$$7 = 2m\,.$$

All we then need to do is to find a number m such that, when multiplied by 2, we obtain the answer 7. This idea of converting a division problem into an equivalent multiplication problem is what we can do for complex numbers.

Since we have already defined multiplication of complex numbers we can apply the idea of "multiplying in order to divide" to the division of complex numbers: $(2+i) \div (3+2i)$ must give some answer in the form $x+iy$. Hence

$$\frac{2+i}{3+2i} = x + iy \qquad \Rightarrow \qquad 2 + i = (3+2i)(x+iy)\,. \tag{*}$$

There are two ways to solve this problem. The first, most obvious way is to expand the RHS and then compare real and imaginary parts. This will require us to solve simulataneous equations. Hence

$$2 + i = 3x - 2y + i(2x + 3y)\,,$$

from which we have $2 = 3x - 2y$ and $1 = 2x + 3y$. Solving this gives $x = 8/13$ and $y = -1/13$. Therefore

$$\frac{2+i}{3+2i} = \frac{8-i}{13}\,.$$

The other, not so obvious way is to multiply both sides of (*) by the conjugate of $3 + 2i$. Hence

$$\begin{aligned}
(2+i)(3-2i) &= (3+2i)(3-2i)(x+iy)\,, && (**)\\
8 - i &= 13(x+iy)\,,\\
\Rightarrow \qquad \frac{8-i}{13} &= x + iy\,.
\end{aligned}$$

This procedure is much faster than the previous one mainly because, in multiplying by the conjugate, we have returned a real number as a factor on the RHS (note that multiplying by conjugates is always a useful tool when wanting to simplify problems). Now notice that (**) can be rewritten as a proper division problem:

$$\frac{2+i}{3+2i}\times\frac{3-2i}{3-2i}=x+iy\,,$$

and this is now how we perform the division of two complex numbers. In other words, given z_1 and z_2 we can perform z_1/z_2 by multiplying top and bottom of the fraction by the conjugate of z_2, i.e.

$$\frac{z_1}{z_2}=\frac{z_1}{z_2}.\frac{z_2^*}{z_2^*}$$

Let us now redo our division according to new approach. Hence

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{2+i}{3+2i},\\ &= \frac{2+i}{3+2i}\times\frac{3-2i}{3-2i}.\\ &= \frac{6-4i+3i-2i^2}{9-6i+6i-4i^2},\\ &= \frac{8-i}{13}.\end{aligned}$$

So in general, if $z_1=a+ib$ and $z_2=c+id$ we have

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{a+ib}{c+id},\\ &= \frac{a+ib}{c+id}\times\frac{c-id}{c-id},\\ &= \frac{ac+bd}{c^2+d^2}+i\frac{bc-ad}{c^2+d^2}.\end{aligned}$$

Multiplying any rational/fractional complex number by the conjugate of the denominator will always result in the denominator being real.

Example 1: Suppose we want to evaluate

$$z = \frac{2-4i}{3+i}.$$

We then proceed as follows:

$$\begin{aligned} z &= \frac{2-4i}{3+i} = \frac{2-4i}{3+i} \times \frac{3-i}{3-i}, \\ &= \frac{(2-4i)(3-i)}{(3+i)(3-i)} = \frac{2-14i}{4}, \\ &= \frac{1}{2} - 7i\,. \end{aligned}$$

Exercise: If $z = x + iy$ show that $1/z = z^*/|z|$.

Example 2: If we want to evaluate

$$z = \frac{(5-i)-(3+7i)}{(4+2i)+(2-3i)},$$

we proceed as follows:

$$\begin{aligned} z &= \frac{(5-i)-(3+7i)}{(4+2i)+(2-3i)}, \\ &= \frac{-2-8i}{6-i}, \\ &= \frac{-2-8i}{6-i} \times \frac{6+i}{6+i} = \frac{(-2-8i)(6+i)}{(6-i)(6+i)}, \\ &= \frac{-4-50i}{7}. \end{aligned}$$

Example 3: To evaluate $w = 2i^6 - (2/i)^3 + 5i^{-5} - 12i$ we use the standard properties of i^2, i^4, etc. Hence we have $w = 2i^4 i^2 - 8/i^3 + 5/i^5 - 12i = -2 - 8/(i^2 i) + 5/(i^4 i) - 12i$. Recall that $1/i = -i$ we now obtain $w = -2 - 8i - 5i - 12i = -2 - 25i$.

Example 4: To find the real and imaginary part of

$$z = \frac{1}{(1+i)(1-2i)},$$

we proceed as follows:

$$\begin{aligned} z &= \frac{1}{(1+i)(1-2i)}, \\ &= \frac{1}{3-i} = \frac{1}{3-i} \times \frac{3+i}{3+i}, \\ &= \frac{3+i}{10}. \end{aligned}$$

Hence $Re(z) = 3/10$ and $Im(z) = 1/10$.

<u>*Example 5:*</u> If $z = x + iy$ what is $Re(1/z)$?

<u>*Solution:*</u>

$$\begin{aligned} \frac{1}{z} &= \frac{1}{x+iy}, \\ &= \frac{1}{x+iy} \times \frac{x-iy}{x-iy}, \\ &= \frac{x-iy}{x^2-y^2}. \end{aligned}$$

Hence $Re(1/z) = x/(x^2 - y^2)$.

<u>*Example 6:*</u> Given that $z = x + iy$ one way we can solve

$$z + 2z^* = \frac{2-i}{1+3i},$$

Is as follows: $z + 2z^* = x + iy + 2x - 2iy = 3x - iy$. Hence we have

$$\begin{aligned} 3x - iy &= \frac{2-i}{1+3i}, \\ &= \frac{2-i}{1+3i} \times \frac{1-3i}{1-3i}, \\ &= \frac{-1-7i}{10}. \end{aligned}$$

Comparing *Re* and *Im* parts we have $x = -1/30$ and $y = 7/10$.

<u>*Exercise:*</u> Solve $z/(1 + z^*) = 3 + 4i$.

Example 7: Note that

$$\frac{1}{(1+i)^2}+\frac{1}{(1-i)^2}=\frac{1}{2i}-\frac{1}{2i}=0\,. \qquad \{*\}$$

Now note that

$$\frac{1}{(1+i)^4}+\frac{1}{(1-i)^4}=\left(\frac{1}{2i}\right)^2+\left(-\frac{1}{2i}\right)^2=\frac{1}{4i^2}+\frac{1}{4i^2}=-\frac{1}{2}.$$

However, if we separately square the terms in {*} we obtain

$$\frac{1}{(1+i)^4}+\frac{1}{(1-i)^4}=\left(\frac{1}{2i}\right)^2-\left(\frac{1}{2i}\right)^2=0$$

which is the wrong answer. To see why we must consider the original rational expression carefully

$$\frac{1}{(1+i)^4}+\frac{1}{(1-i)^4}=\frac{1}{[(1+i)^2]^2}+\frac{1}{[(1+i)^2]^2}=\left(\frac{1}{2i}\right)^2+\left(-\frac{1}{2i}\right)^2,$$

noticing in general that $-(i)^2\neq(-i)^2$.

Example 8: If $z=x+iy$ and

$$\frac{|z-3|}{|z+3|}=2\,,$$

show that this gives to the equation of a circle centre $(-5,0)$, radius 4.

Solution:

$|z-3|=|x-3+iy|=\sqrt{(x-3)^2+y^2}$, and $|z+3|=\ |x+3+iy|=\sqrt{(x+3)^2+y^2}$. Hence

$$\frac{|z-3|}{|z+3|}=2$$

implies

$$\sqrt{(x-3)^2+y^2}=2\sqrt{(x+3)^2+y^2}\,,$$

from which

$$(x-3)^2+y^2=4(x+3)^2+y^2\,.$$

Expanding and simplifying we obtain $x^2+y^2+10x+9=0$. Completing the square we end up with $(x+5)^2-25+y^2+9=0$ implying $(x+5)^2+y^2=16$. This is the equation of a circle centre $(-5,0)$, radius 4. Also, knowing that $|z|$ is given by $\sqrt{x^2+y^2}$ we can express the equation of the circle as $|z+5|=4$.

Example 9: Let $z = x + iy$. Express $|z - 1 - 3i|^2 = 4$ and $|z + z^*| = 1$ in terms of x and y. Describe the geometric meaning of these expressions.

Solution

For the first expression we have $|z - 1 - 3i|^2 = |x + iy - 1 - 3i|^2 = 4$. Hence $|(x-1) + i(y-3)|^2 = 4$. Since the modulus involves a square root which we are then going to square we end up with $|z - 1 - 3i|^2 = (x-1)^2 + (y-3)^2 = 4$. This represents a circle of centre $(1, 3)$ and radius 2.

For the second expression we have $|z + z^*| = |x + iy + (x - iy)| = |2x| = 2x = 1$. Hence $x = 1/2$ and this represents the vertical line through $x = ½$.

1.8.2 The geometric effect of division on complex numbers

How are we going to interpret the geometric effect of this division? Well, as with multiplication, the division of z_1 by z_2 will rotated and scale z_1. Then, if the process of multiplication makes z_1 rotate towards and beyond z_2 then it seems logical to assume that the process of division will make z_1 rotate away from z_2. Similarly if the process of multiplication generally results in z_1 being stretched then the process of division will generally result in z_1 shrinking. This is indeed the case, and we now need to know the extent of this rotation and scaling.

To do this let us view division as a *multiplication-by-conjugate*. Remember that, in performing $z = (2 + i)/(3 + 2i)$, which we do

$$z = \frac{2 + i}{3 + 2i} \times \frac{3 - 2i}{3 - 2i}.$$

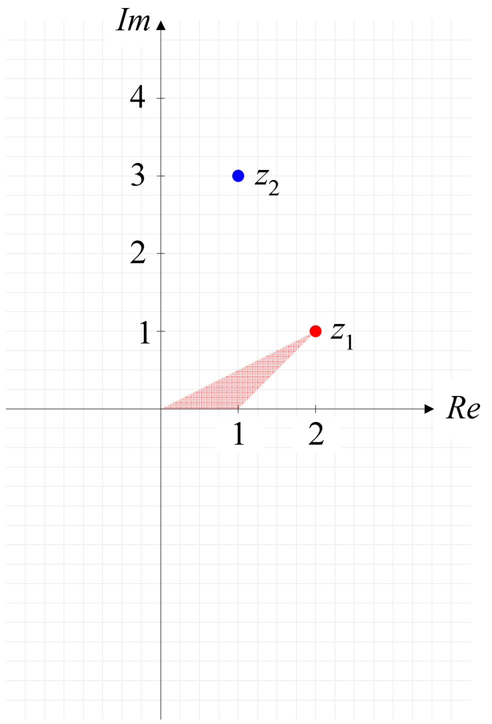
Let us now look at this as

$$z = \frac{(2 + i)(3 - 2i)}{(3 + 2i)(3 - 2i)},$$

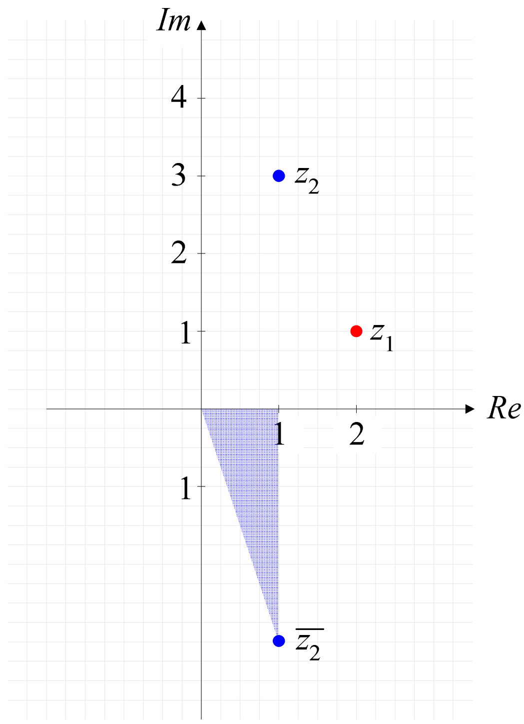
and consider the effect of the numerator and denominator separatetly. Looking at the denominator first, we know that the product of conjugates gives us a real number, say m. So what we are really doing is scaling the numerator by $1/m$.

We are therefore left with understanding the geometric effect of multiplying $(2 + i)$ by $(3 - 2i)$, which is the conjugate of the divisor. To understand this effect we will use the same approach as we used in multiplication, but with one important change, namely we will rotate the triangle formed from z_1 so that its base meets the line to the *conjugate* of z_2.

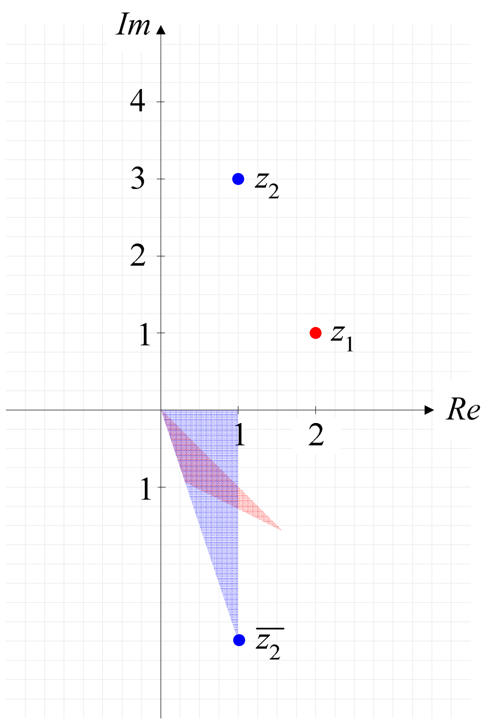
Therefore, let us consider the geometric effect of dividing $z_1 = 2 + i$ by $z_2 = 1 + 3i$. Our process will therefore be: i) rotate the triangle formed by z_1 so that its base lies on the line to $\overline{z_2}$; ii) shrink the base of the rotated z_1 triangle by the amount given by the hypotenuse of the $\overline{z_2}$ triangle, as illustrated below:



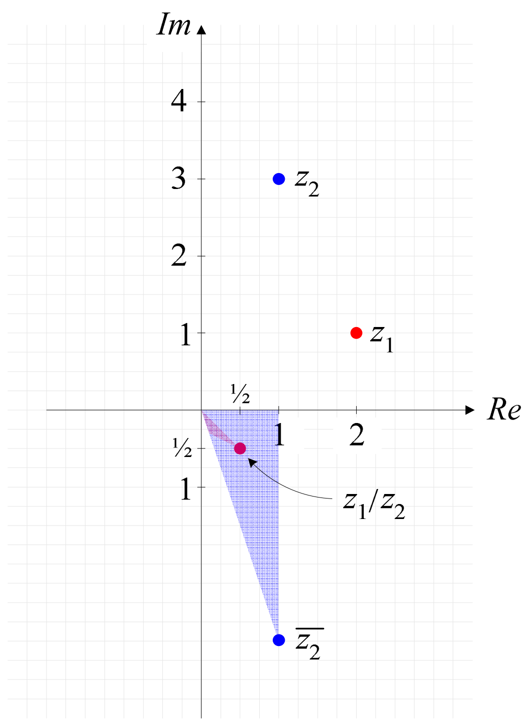
The triangle formed by z_1



The triangle formed by the conjugate of $\overline{z_2}$



Alligining the base of the z_1 triangle to the hypotenuse of $\overline{z_2}$



Shrinking the base of the alligned z_1 triangle by a factor of $\overline{z_2}$

In this case it is difficult to see visually the extent to which the base of triangle z_1 should be shrunk, but the algebra of division determines this for us. The above can be shown to be the case by considering the components of the arithmetic of division, as was illustrated for the case of multiplication in section 1.7.2. This is left as an exercise.

The geometric effect of division of complex numbers can then be summarised as follows:

given z_1/z_2 we rotate the complex number $z_1 z_2$ by $\pm\pi/2$ radians,
and then scale this by a factor given by $z_2.\overline{z_2}$

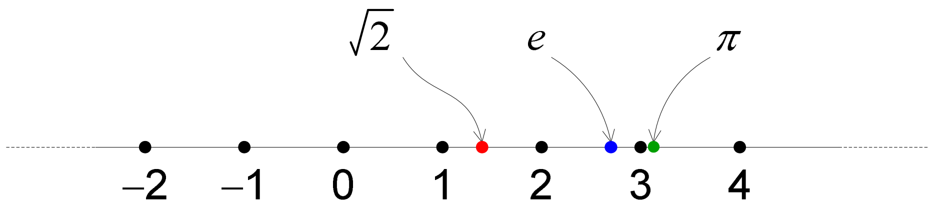
<u>*Exercise:*</u> Let $z = a + ib$. What is the geometric relationship between z and z^{-1}?

1.9 On the ordering of complex numbers

The following is adapted from "Ordering complex numbers… Not", David Angell, *Parabola*, Volume 43, Issue 2 (2007) hosted at the University of New South Wales, and from "A new approach to ordering complex numbers", D. K. Yadav, *International Journal of Math. Sci. & Engg. Appls. (IJMSEA)*, Vol. 2 No. III (2008), pp. 211-223.

1.9.1 Some attempts to order complex numbers, and why they fail

As we know, the real numbers are arranged in order, an order which can be visualised by the number line. This ordering is unique, such that certain numbers always come before or after certain other numbers, and do so in only one way. So when we say that $2 < 3$ this is the only ordering possible between the numbers 2 and 3. This ordering can be said to carry-over. For example if $2 < 5$ and $5 < 9$ then $2 < 9$. This is true for all real numbers in general: if $a < b$ and $b < c$ then $a < c$. This means that the set of real numbers can informally be said to be *uniquely uniform in ordering*.



Beyond this we also have properties of ordered numbers which apply when we perform arithmetic operation on a given ordering: if a and b are two real numbers such that $a < b$ then for a third real number c we have $a + c < b + c$. Similarly, for any non-negative number c we have $ac \leq bc$.

The question now is, Is it possible to order the complex numbers in a unique way and preserve this ordering under addition and multiplication? If it is possible, then we can place all complex numbers in a unique order, and talk about one complex number being less than or greater than another complex number. The short answer to this question is, No, and the rest of this section is designed to provide some conceptual understanding of why this is so.

Before we can talk about ordering we first need to talk about the equality of number. This may seem redundant since, if two real numbers x and y are such that $x \nless y$ and $y \nless x$, then it seems obvious that the only possibility left is that $x = y$ (this leads us to the law of trichotomy, i.e. given two numbers x and y then only one of the following is true: $x < y$ or $x = y$ or $x > y$).

The question now is, Can we set up a similar definition of equality for complex numbers? Yes. But, given that complex numbers consist of two components, it is not obvious what two equal complex numbers should look like. So we need an explicit defintion for this. As such, we say that

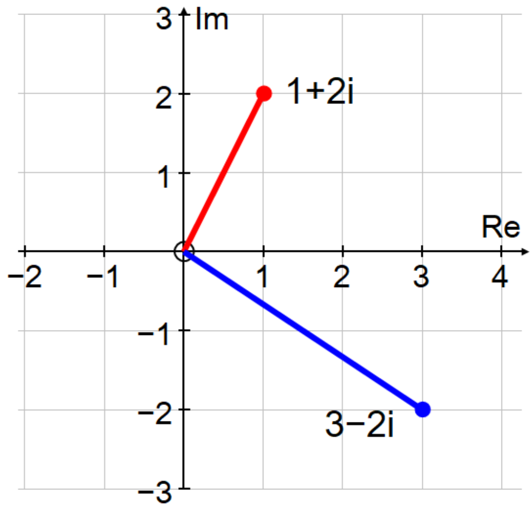
two complex numbers $a + ib$ and $c + id$ are equal when $a = c$ and $b = d$.

This can be considered a definition of the equality of complex numbers. So if $2 - 3i = a + ib$ we know that $a = 2$ and $b = -3$.

Now, when we say $x < y$, for any two distinct real numbers x and y, we are comparing a single value, x, with another single value, y, so there is no ambiguity as to which value is the smaller. However, when we want to discuss the ordering of $a + ib$ and $c + id$, for $a, b, c, d \in \mathbb{R}$, we have to consider and compare two values per complex number. How do we do this?

For example, what does it mean for $1 + 2i < 3 - 2i$, if this is at all possible? Is $1 + 2i < 3 - 2i$ because $1 < 3$? What about $3 - 2i < 1 + 2i$? Here $3 \nless 1$ but $-2 < 2$. So is $3 - 2i < 1 + 2i$ because of this latter criterion? Or is there another way of comparing these two complex numbers such that $1 + 2i$ is definitively less than $3 - 2i$?

The problem we have encountered here is due to the lack of uniqueness in which parts of a complex number to use for comparison. And if we do decide to define $a + ib < c + id$ when $a < c$ the question is, Why? What makes $a < c$ the necessary criterion for deciding on "less than"?



How to we compare the sizes of these two complex numbers?

So does this mean that complex numbers cannot be ordered? No. There is a way of ordering complex numbers, and it is very easy to do so. We have already shown two ways in which complex numbers can be ordered: we either define "<" for two complex numbers on the basis of $a < c$ or on the basis of $b < d$. But are these criteria sufficient for us to be able to define a consistent ordering of the complex numbers such that this ordering is preserved under addition and multiplication in the same way that it is preserved for real number (as illustrated earlier)? This is what we will now look into.

Therefore, let $a + ib$ and $c + id$ be two distinct complex numbers. One way to define an ordering of complex numbers is as follows:

<u>*Definition 1:*</u> $a + ib < c + id$ if $a < c$ or $b < d$.

Hence $2 + 5i < 9 - i$. Similarly $9 - i < 1 + 3i$. The question now is, is $2 + 5i < 1 + 3i$? No since $2 \nless 1$ and $5 \nless 3$. So there is no uniformity of ordering here, and definition 1 does not work to help us order complex numbers in a uniform way.

Let us therefore try another definition which overcomes this problem. For example, we could define an ordering of complex numbers as follows

<u>*Definition 2:*</u> $a + ib < c + id$ if $a + b < c + d$

Using the example above we now have that $2 + 5i < 9 - i$ since $2 + 5 < 9 - 1$. Similarly $9 - i < 8 + 7i$. Furthermore, if we are to keep the uniformity of ordering that exists for real numbers then $2 + 5i < 9 - i$ and $9 - i < 8 + 7i$ has to imply $2 + 5i < 8 + 7i$, which is true by definition 2. So it seems that definition 2 satisfies the property of uniformity of ordering, but does it satisfy this property in general? In other words, if $a + ib < c + id$ and $c + id < e + if$ will it be true that $a + ib < e + if$ by definition 2? Yes, by the usual definition of "<" for real numbers.

However, we now encounter another problem: how do we compare $2 + 5i$ and $5 + 2i$? Well, by definition 2 we have $2 + 5 = 5 + 2$, so neither number is less than the other number. By the law of trichotomy this must imply that these two numbers are equal. But it is obvious that $2 + 5i \neq 5 + 2i$. Hence definition 2 does not work to help us uniquely order complex numbers in a uniform way, such that equality holds true.

Let us therefore try a third definition:

Definition 3: $a + ib < c + id$ if $a < c$ and $b < d$

According to this definition $2 + 5i < 8 + 6i$ and $8 + 6i < 9 + 7i$, so definition 3 allows for uniformity of ordering, and we also overcome the problem caused by definition 2 in that $2 + 5i \neq 8 + 6i \neq 9 + 7i$. However, $2 + 5i \nless 9 - i$ and $9 - i \nless 2 + 5i$, so in this case we cannot tell which of $9 - i$ and $2 + 5i$ is the smaller. Hence definition 3 does not work to help us uniquely order complex numbers in a uniform way.

Let us therefore define another ordering of complex numbers:

Definition 4: $a + ib < c + id$ if either $a < c$, or $a = c$ and $b < d$

Then, $2 + 5i < 9 - i$ and $9 - i < 9 + 7i$. In this case it is also true that $2 + 5i < 9 + 7i$ so the property of uniform ordering stands, and there is no problem about deciding whether one complex number is less than another complex numbers. So it seems as if we have solved the problem of the ordering of complex numbers.

1.9.2 Trying to preserve the ordering of complex numbers under addition and multiplication

Returning to the case of real numbers we saw at the beginning of this section that the following property holds: if a and b are two real numbers such that $a < b$ then for a third real number c we have $a + c < b + c$. Similarly, for any $c \geq 0$, $ac \leq bc$. So, for ordering to be of any value we must be able to perform the operations of addition and multiplication on values in order, whilst keeping these values appropriately ordered. For example, $2 < 5$ implies $2 + 1 < 5 + 1$. Or $7 < 11$ implies $7 \times 2 < 11 \times 2$. Or $-3 < 1$ implies $-3 \times -2 > 1 \times -2$.

Does this ability to perform arithmetic hold for complex numbers ordered according to definition 4? No. To see why consider $0 < 1 + 2i$. We know that $2 + 5i < 9 - i$ so we should have $(2 + 5i)(1 - 2i) < (9 - i)(1 - 2i)$. But this latter inequality simplifies to $12 + i < 7 - 19i$ which is not true by definition 4. So definition 4 does not work to help us uniquely order complex numbers in a uniform way.

1.9.3 Another attempt to order complex numbers, and why it fails

Our definitions so far have been based on the Cartesian form of a complex number. What if we choose to define an ordering based on the polar form of a complex number, $r(\cos\theta + i\sin\theta)$, where, for $z = x + iy$, we have $r = \sqrt{x + iy}$ and $\theta = \tan^{-1}(y/x)$? Ultimately we could adopt a similar definition to that of definition 4 for the ordering of complex numbers:

Definition 5: $a + ib \le c + id$ if

- either $\sqrt{a^2 + b^2} < \sqrt{c^2 + d^2}$;
- or $\sqrt{a^2 + b^2} = \sqrt{c^2 + d^2}$ and $\tan^{-1}(b/a) < \tan^{-1}(d/c)$

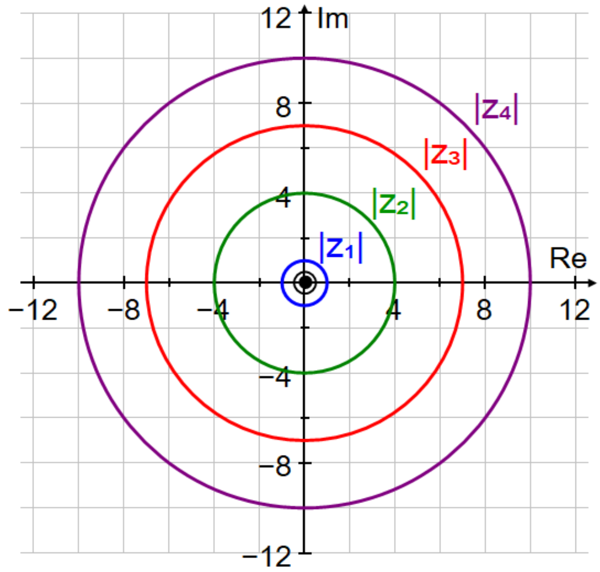
Then $1 + 2i < 2 + 3i$. Also, $\sqrt{3} + i < \sqrt{2} + i\sqrt{2}$ and $\sqrt{2} + i\sqrt{2} < 1 + i\sqrt{3}$. Is there a uniformity of ordering in this last example? Yes since $\sqrt{3} + i < 1 + i\sqrt{3}$, but does this uniformity of ordering work in general, and is such an ordering preserved under the operations of addition and multiplication? No. The easy way to realise this is that every complex number in polar form can be transformed into its equivalent Cartesian form, and we have seen that complex numbers in Cartesian form do not preserve ordering under addition and/or multiplication.

1.9.4 Yet another attempt to order complex numbers, and why it fails

There is an alternative way of conceiving of the ordering of complex numbers, but this will require us to redefine what it means for two complex numbers to be equal. As previously mentioned two complex numbers $a + ib$ and $c + id$ are equal when $a = c$ and $b = d$. This is the current state of the mathematical definition of equality of complex numbers, and is the one always referred to.

But by changing this definition we will be able to develop a unique uniform ordering of complex numbers which is preserved under the operations of addition and multiplication.

To begin with we need to understand the geometric meaning of an expression such as $|z| = 1$. If $z = x + iy$ we have $\sqrt{x^2 + y^2} = 1$, implying that $x^2 + y^2 = 1$. This is the equation of a circle of centre $(0, 0)$ and radius 1. Similarly, $|z| = 2$ a circle of centre $(0, 0)$ and radius 2, etc. and in general $|z| = k$ (where $k \in \mathbb{R}$) is a circle of centre $(0, 0)$ and radius k. This situation is illustrated below:



From this diagram we see that $0 \leq |z_1| \leq |z_2| \leq |z_3| \leq |z_4|$, and in general we can extend this to be $0 \leq |z_1| \leq |z_2| \leq |z_3| \leq |z_4| \leq \cdots \leq |z_n|$ where $|z_i| = r_i$ where r_i is a real number such that $0 \leq r_i < n$, and $n \to \infty$.

However, the same problem of equality of complex numbers occurs here as was discussed above. Consider the following four complex numbers of $z_1 = 3 + 4i$, $z_2 = -3 + 4i$. It is clear from our earlier work that we cannot tell if $z_1 < z_2$ or $z_2 < z_1$ or $z_1 = z_2$. But what we do know is that they all lie on the same circle given by $x^2 + y^2 = 25$, i.e. $|z| = 5$. Therefore, in order to get around the problem of ordering we can define a new equality of complex numbers, called *equi-radii complex numbers*, such that two complex are equal if they lie on the circumference of the same circle. Hence $z_1 = 3 + 4i$ and $z_2 = -3 + 4i$ are equal in this sense. Similarly, $w_1 = \sqrt{3} + i$ and $w_2 = \sqrt{2} + i\sqrt{2}$ are equal in this sense.

To emphasise: what we are doing here is to redefine the concept of equality of complex numbers. Here the equality is based on complex numbers lying on the same circle given by $|z_i| = r_i$, and not on the equality of the *Re* and *Im* components of complex numbers.

So we can state another defintion for the ordering of complex number:

Definition 6

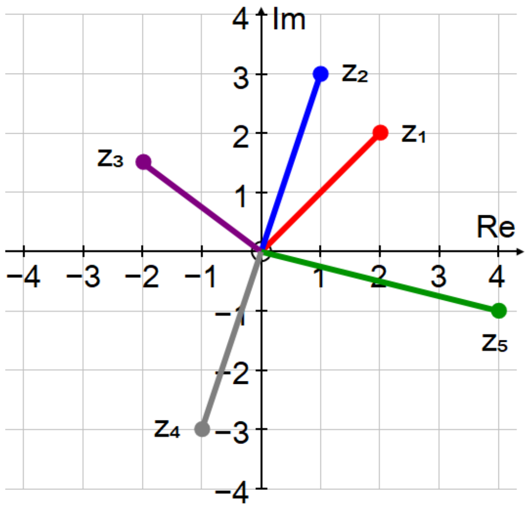
Equality: $z_1 = z_2$ if and only if $|z_1| = |z_2|$

Ordering: $z_1 < z_2$ if and only if $|z_1| < |z_2|$

An alternative, or derived, law of trichotomy, called the D-law of trichotomy by Yadav, can now be stated: for any two complex numbers z and w exactly one of $z \sqsupset w$ or $z \sqsubset w$ or $z \,\square\, w$ is true according to $|z| < |w|$, $|z| > |w|$, or $|z| = |w|$ respectively (where "$\sqsupset$" is read as "less than", "$\sqsubset$" is read as "greater than", and "$\square$" is read as "equals").

Example 1: The complex numbers $z_1 = 1 + i$, $z_2 = 2 - i$, $z_3 = 2 + i$, and $z_4 = 3 + i$ can be ordered as $|z_1| < |z_2| = |z_3| < |z_4|$, i.e. as $z_1 \sqsupset z_2 \,\square\, z_3 \sqsupset z_4$.

Example 2: For the four complex numbers illustrated below we see that $z_1 \sqsupset z_3 \sqsupset z_2 \,\square\, z_4 \sqsupset z_5$.



This can be seen visually by the fact that, since each vector acts as the radius of a circle from $(0, 0)$, the length of the shortest vector (and therefore, the smallest radius) is that of z_1, followed by z_3, followed by z_2 and z_4 which have equal length, followed by z_5.

Continuing with our analysis, since r_i is a real number we can completely order all complex numbers according to their moduli, as well have these numbers obey the law of trichotomy, as well as preserve such ordering under addition and multiplication. As such we can state the following laws of ordering, the proofs of which are left as an exercise:

- *Linear ordering*: If $z < w$ then there exists a complex number v such t$w = z + v$;
- *Law of trichotomy*: For any two complex numbers z and w only one of the following is possible: $z < w$, $z = w$, or $z > w$;
- *Law of transitivity*: If $z < w$ and $w < v$ then $z < v$;
- *Law of addition*: If $z < w$ then $z + v < w + v$ for any complex number v;
- *Law of multiplication*: If $z < w$ then $vz < vw$ for any complex number $v > 0$;

(note that the above is not a complete set of laws necessary to define ordering. For a complete set of laws that define ordering, such as for real numbers, see any book on real anlysis).

1.9.5 Conclusion

Does this mean we have overcome all the problems of ordering complex numbers uniquely such that ordering is preserved under addition and multiplication? No. We still have a problem, and it is to do with our definition of the equality of complex numbers. The fact of the matter is that equality of two complex numbers means that both numbers are really one and the same number, and occupy the same unique location on the Argand diagram. So, for $2 + 3i$ to be equal to $a + bi$, and therefore to define one and only one location in the Argand diagram, we must have $a = 2$ and $b = 3$. So there is only one geometric location at which we can draw $(2, 3)$.

But if we define equality of complex numbers via the equality of their moduli, i.e. $z_1 = \sqrt{3} + i$ and $z_2 = -\sqrt{3} + i$ are equal because $|z_1| = |z_2|$ then we have lost the property of uniqueness of geometric/spatial location since it is clear that z_1 and z_2 lie in different locations in the Argand diagram. This is because, geometrically speaking, the position of a number $a + ib$ in the Argand diagram is illustrated as a point, not as a circle or any other path, and things like $|z_1| = |z_2|$ define a path, not a point.

So, however we try to adjust our definitions for the ordering of two complex numbers $z_1 = a + ib$ and $z_2 = c + id$, either as (for example)

Equality: $z_1 = z_2$ if and only if $a = c$ and $b = d$

Ordering: $z_1 < z_2$

if and only if $a < c$ or $a = c$ and $b < d$,

or

if and only if $|z_1| < |z_2|$ or $|z_1| = |z_2|$ and $\arg(z_1) < \arg(z_2)$

or as

Equality: $z_1 = z_2$ if and only if $|z_1| = |z_2|$

Ordering: $z_1 < z_2$ if and only if $|z_1| < |z_2|$

we end up losing something. If we adopt the former group as our definitions we lose the ability to order complex numbers. If we adopt the latter group as our definition we lose the uniqueness of equality of complex numbers.

And there is yet another problem with definition 6: Why choose equality of complex numbers based on their moduli? We could very well have defined equality of complex numbers as equality of arguments. And since $\arg(z)$ is a real number we would also have obtained a uniform ordering. For example, $1 + i$, $2 + 2i$, $3 + 3i$ etc. are all equal complex numbers since they all have the same argument of $\arg(z) = \pi/4$. However, if $w_1 = \sqrt{2} + i\sqrt{2}$ and $w_2 = 1 + i\sqrt{3}$ then w_1 is less than w_2 since $\arg(w_1) = \pi/4 < \arg(w_2) = 2\pi/3$.

A uniform ordering of complex numbers could then be developed on this basis, and the relevant law of trichotomy could be developed, as well as the preservation of ordering under addition and multiplication.

However, this again this brings up the issue of the non-uniqueness of the definition of equality: which one do we choose? The one based on moduli of complex numbers or the one based on the argument of the complex numbers?

Ultimately, given the requirements for the definition of equality of numbers, the law of trichotomy, and the preserving of ordering under addition and multiplication (as well as other necessary axioms not discussed here), it is not possible to define a unique uniform ordering of complex numbers which also preserved ordering under addition and multiplication.

1.10 The quadratic formula for a quadratic equation whose coefficients are complex numbers

We have seen that the way we do arithmetic on real numbers does not completely transfer to the way we do it on complex numbers. It is this problem that not everything we do on real numbers can be done in the same way on complex numbers which leads us to realising that every single operation and algebra done on real numbers has to be redefined for complex numbers.

As such, we cannot assume that an arithmetic, a formula or a function which works on real numbers will work in the same way on complex numbers. Furthermore, there are cases where an arithmetic, a formula or a function is not defined for real numbers x, but is defined for complex numbers z. For example, if x is a real number such that $x > 0$ then $\log x$ exists. But if $x \leq 0$ then $\log x$ does not exists. However, we will see later on that the function $\log z$ does exists for any complex number, including negative real numbers. In other words we will be able to find $\log(-1)$ when -1 is seen as a complex number (more on this later).

Let us therefore consider one basic formula, that of the quadratic formula for solving the equation $ax^2 + bx + c = 0$, where a, b, and c are real numbers and where $x \in \mathbb{R}$. We know that the roots are given by $x = \left(-b \pm \sqrt{b^2 - 4ac}\right)/(2a)$. The question now is, Does this formula hold when our variable is complex and when the coefficients are complex? We don't know, so we have to prove it.

Therefore, let z be a complex variable such that $az^2 + bz + c = 0$, and let the coefficients a, b, c be complex numbers. In deriving a formula for the roots z_1 and z_2 we will go through all the usual steps necessary, with the following issue in mind: we will need to make sure we correctly use all the previously defined rules of arithmetic on complex numbers (not the rules of real arithmetic we are used to).

So, given

$$az^2 + bz + c = 0$$

where $a, b, c \in \mathbb{C}$, we wish to divide by a. Is this possible given that a is a complex number? Yes since division by complex numbers is defined. Hence we have

$$z^2 + \frac{b}{a}z + \frac{c}{a} = 0\,,$$

from which we have

$$z^2 + \frac{b}{a}z = -\frac{c}{a}\,,$$

where this last operation of subtraction is also defined for complex numbers. The next step we want to be able to do is that of completing the square, and then solving for z.

Are these arithmetic processes valid for complex numbers? Yes, because they simply involve multiplication, subtraction and square rooting, all of which are defined for complex numbers. Hence

$$\left(z+\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}=-\frac{c}{a},$$

$$\left(z+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a}=\frac{b^2-4ac}{4a^2}.$$

Now we arrive at a very formal part of the proof. At this point we might be tempted to take the square root of both sides. But since complex numbers are new objects to us, and with our newly gained understanding about how to treat and analyse new objects, we cannot assume that square rooting will work directly on complex numbers a, b, and c as they do on real numbers (remember that rooting is, for the moment, only defined for real numbers).

So how do we get around this problem? Well, we do know how to square complex numbers, since we have already defined a way of doing this. We will therefore adopt the approach of specifying terms as squares rather than as square roots (this is the same approach we adopted in section 1.7.4 when we calculated the square root of a complex number).

Formally speaking our aim is therefore to find solutions to

$$w^2=\frac{b^2-4ac}{4a^2}, \tag{20}$$

where $w=z+b/(2a)$. The process of solving quadratic equations with complex coefficients in this way is therefore a two-step process:

- first use (20), and simplify this answer.
- then
    - if the answer is a real number, take square roots as usual to get w, then use $w=z+b/(2a)$ to find z;

  or
    - if the answer is complex, take the square root as in section 1.7.4, then use $w=z+b/(2a)$ to find z.

For all practical purposes the expression

$$z = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}\,, \tag{21}$$

(where a, b, c are complex numbers) found in most books will work. It is basically a short hand for the two-stage process above. But, at the formal level, it has to be understood that the "$\sqrt{}$" symbol is only to be used for real numbers. When we do get to defining the rooting process for complex numbers we will use the symbols of rational powers, i.e. "½", etc.

The same requirement will apply for other formulae and functions valid in $\mathbb{C}$. In other words, we will need to define functions such as $f(z) = e^z$ or $f(z) = \log z$, as well as define (amongst other things) differentiation and intregration when performed on functions with complex variables.

Examples

1) Suppose that we want to find the roots of $z^2 + iz - 2 = 0$, where $z = x + iy$. Using the formal approach involving equation (20) we have

$$w^2 = \frac{i^2 - 4(1)(-2)}{4(1)^2} = \frac{7}{4}\,.$$

Since $7/4$ is real we can take roots directly, which gives us $w = \pm\sqrt{7}/2$. From $w = z + b/(2a) = z + i/2$ we obtain z to be

$$z = \tfrac{1}{2}\left(\sqrt{7} - i\right) \text{ and } z = -\tfrac{1}{2}\left(\sqrt{7} + i\right).$$

Note that $z^2 + iz - 2 = 0$ can now be factorised as

$$\left(z - \left(\frac{\sqrt{7}}{2} - \frac{i}{2}\right)\right)\left(z - \left(-\frac{\sqrt{7}}{2} - \frac{i}{2}\right)\right) = 0\,.$$

2) Let $z = x + iy$. Find the roots of $z^2 + (2i - 3)z + 5 - i = 0$. Hence factorise this quadratic.

Solution: By the standard quadratic formula we have

$$z = \frac{-(2i - 3) \pm \sqrt{(2i - 3)^2 - 4(5 - i)}}{2} = \frac{3 - 2i \pm \sqrt{-15 - 8i}}{2}\,.$$

We now need to find $\sqrt{-15-8i}$. Hence

$$\sqrt{-15-8i} = a + ib\,.$$

Squaring both sides gives

$$\begin{aligned} -15-8i &= (a+ib)^2\,, \\ &= (a^2-b^2) + i(2ab)\,. \end{aligned}$$

We now compare *Re* and *Im* coefficients to get

$$-15 = a^2 - b^2\,, \tag{22}$$

$$\text{and} \qquad -8 = 2ab \qquad \text{implying} \qquad -4 = ab\,. \tag{23}$$

Substituting (23) into (22) we obtain

$$-15 = a^2 - \left(\frac{-4}{a}\right)^2\,,$$

which simplifes to

$$a^4 + 15a^2 - 16 = 0\,. \tag{24}$$

This can be factorised as

$$(a^2+16)(a^2-1) = 0\,,$$

implying that $a^2 + 16 = 0$ or $a^2 - 1 = 0$. However, $a^2 + 16 = 0$ gives negative a^2, leading to a being complex. But a is a real number so the negative square root case is not valid. Therefore $a^2 - 1 = 0$ is the only valid factor. Hence $a = \pm 1$,

$$a = \pm 1\,, \tag{25}$$

$$b = \mp 4\,. \tag{26}$$

But by (23) we know that ab is negative, so a and b must have opposite signs. Therefore $a = 3$ and $b = -4$, or $a = -1$ or $b = 4$, and the two roots of $1 - 4i$ are $-1 + 4i$. This then gives us

$$z = \frac{3 - 2i \pm (1-4i)}{2} = 2 - 3i, 1 + i\,,$$

from which we write

$$\big(z - (2-3i)\big)\big(z - (1+i)\big) = 0\,.$$

3) Find the roots of $z^2 + (1+2i)z - 2 + 3i = 0$ given that $z = x + iy$.

<u>*Solution*</u>: By the standard quadratic formula (21) we have

$$z = \frac{-(1+2i) \pm \sqrt{(1+2i)^2 - 4(-2+3i)}}{2} = \frac{-(1+2i) \pm \sqrt{5-8i}}{2}.$$

We now need to find $\sqrt{5-8i}$. Hence

$$\sqrt{5-8i} = a + ib\,.$$

Squaring both sides gives

$$\begin{aligned} 5 - 8i &= (a+ib)^2\,, \\ &= (a^2 - b^2) + i(2ab)\,. \end{aligned}$$

We now compare *Re* and *Im* coefficients to get

$$5 = a^2 - b^2\,, \tag{27}$$

$$\text{and} \qquad -8 = 2ab \qquad \text{implying} \qquad -4 = ab\,. \tag{28}$$

We could now substitute (28) into (27) for b, but instead we will create a third equation based on the modulus of z. Therefore, since $\sqrt{5-8i} = a + ib$, we have $5 - 8i = (a+ib)^2 = (a^2 - b^2) + i(2ab)$. Taking the modulus of both sides gives

$$\sqrt{5^2 + (-8)^2} = \sqrt{(a^2 - b^2)^2 + (2ab)^2} \text{ implying } \sqrt{89} = a^2 + b^2 \tag{29}$$

Adding and subtracting (27) and (29) appropriately and take the square root we get

$$a = \pm\sqrt{\frac{1}{2}\left(5 + \sqrt{89}\right)}\,,$$

and

$$b = \pm\sqrt{\frac{1}{2}\left(-5 + \sqrt{89}\right)}\,.$$

By (28) the signs of a and b are opposite, so the roots of $5 - 8i$ are $a = \frac{1}{\sqrt{2}}\sqrt{\left(5+\sqrt{89}\right)}$ and $b = -\frac{1}{\sqrt{2}}\sqrt{\left(-5+\sqrt{89}\right)}$, or $a = -\frac{1}{\sqrt{2}}\sqrt{\left(5+\sqrt{89}\right)}$ and $b = \frac{1}{\sqrt{2}}\sqrt{\left(-5+\sqrt{89}\right)}$.

Exercises: Find the roots of i) $z^2 - (1+i)z + 6 - 17i = 0$, ii) $z^2 - 2z + 1 - 2i = 0$, iii) $iz^2 - z + i = 0$.

4) Given that $z_1 = -\frac{3}{2} + \frac{5}{2}i$ is a root of $f(z) = 4z^2 + 12z + 34 = 0$, factorise $f(z)$.

Solution: The quadratic has real coefficients, therefore $\overline{z_1}$ is also a root of $f(z)$. Hence

$$4z^2 + 12z + 34 = \left(z - \left(-\frac{3}{2} + \frac{5}{2}i\right)\right)\left(z - \left(-\frac{3}{2} - \frac{5}{2}i\right)\right) = 0\,.$$

5) To find a quadratic $f(z) = az^2 + bz + z = 0$ such that one root of $f(z)$ is $2 - i$ we can proceed as follows: if $2 - i$ is a root then $2 + i$ is also a root when $a, b, c \in \mathbb{R}$. Hence

$$\big(z - (2 - i)\big)\big(z - (2 + i)\big) = 0\,,$$

giving

$$\begin{aligned} z^2 - z[(2 - i) + (2 + i)] + 5 &= 0\,, \\ \Longrightarrow \qquad z^2 - 4z + 5 &= 0\,. \end{aligned}$$

Is this quadratic unique in having the root $2 - i$? No. Firstly there may be other quadratics with real coefficients which have $2 - i$ as a root. Secondly, there may be quadratics with complex coefficients which have $2 - i$ as a root. To see this latter possibility let $z_2 = x + iy$ be another root of $f(z)$ such that $z_2 \neq \overline{z_1}$. Then

$$\begin{aligned} az^2 + bz + c &= \big(z - (2 - i)\big)\big(z - (x + iy)\big)\,, \\ &= z^2 - z[(2 - i) + (x + iy)] + (2x + y) + i(2y - x)\,, \\ &= z^2 - z[(2 + x) + i(y - 1)] + (2x + y) + i(2y - x)\,. \end{aligned}$$

Comparing *Re* and *Im* coefficients we have

$$\begin{aligned} a &= 1\,, \\ b &= -2 - x - i(y - 1)\,, \\ c &= 2x + y + i(2y - x)\,. \end{aligned}$$

We are now free to choose any value for x and y to solve for b and c. To make things easy choose $x = -3$ and $y = 2$ in order to get a simple number for b.

Hence $b = 1 - i$ and $c = -4 + 7i$, and we have

$$f(z) = z^2 + (1 - i)z - 4 + 7i = 0$$

as a quadratic equation having one root to be $2 - i$. We can then confirm that $f(2 - i) = 0$.

<u>*Exercise:*</u> For each of the following roots, find a quadratic $f(z) = az^2 + bz + z = 0$ such that $a, b, c \in \mathbb{R}$ and $a, b, c \in \mathbb{C}$:

i) $z_1 = 1 + i$ ii) $z_1 = 1 - 2i$ iii) $z_1 = -1 + 7i/5$

1.11 The Polar form of a complex number

1.11.1 The polar form of a complex number

We saw in section 1.4 that for a complex number $z = x + iy$, we have $x = r\cos\theta$ and $y = r\sin\theta$. As a result of this we are in a position to express z in an alternative form, namely the *modulus-argument form* or *polar form*. Hence $z = x + iy$ becomes

$$z = r(\cos\theta + i\sin\theta)\,. \tag{30}$$

Expression (30) now allows us to convert any complex number from Cartesian form to polar form. Since $x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$ then $|z| = r = \sqrt{x^2 + y^2}$. Also since $y/x = \sin\theta\,/\,\cos\theta$ we have $\theta = \tan^{-1}(y/x)$.

For example,

i) if $z = 1 + i$ then $r = |z| = \sqrt{1^2 + 1^2} = \sqrt{2}$, and $\theta = \arg(z) = \tan^{-1} 1/1 = \pi/4$. Hence $z = \sqrt{2}(\cos\pi/4 + i\sin\pi/4)$;

ii) if $z = \sqrt{3} - i$ then $r = |z| = \sqrt{\left(\sqrt{3}\right)^2 + (-1)^2} = 2$, and $\theta = \arg(z) = \tan^{-1} -1/\sqrt{3} = -\pi/6$. Hence $z = 2(\cos(-\pi/6) + i\sin(-\pi/6)) = 2(\cos\pi/6 - i\sin\pi/6)$.

iii) if $z = 1$ then $r = |z| = \sqrt{1^2 + 0^2} = 1$ and $\theta = \arg(z) = \tan^{-1}(0/1) = 0$. Hence $z = \cos 0 + i\sin 0$.

iv) if $z = -i$ then $r = |z| = \sqrt{0^2 + (-1)^2} = 1$ and $\theta = \arg(z) = \tan^{-1}(-1/0) = -\pi/2$. Hence $z = \cos(-\pi/2) + i\sin(-\pi/2) = \cos\pi/2 - i\sin\pi/2$.

Other examples

1) For the pair of complex numbers $z_1 = 10 + 8i$, $z_2 = 11 - 6i$ which is closest to the origin? Which has rotated by the least amount in absolute terms? Which is closest to $z_3 = 1 + i$?

Solution

Distance to the origin is given by $r = |z|$ and rotation is w.r.t. to the positive real axis and is given by $\theta = \arg z$. Hence for distance we have $|z_1| = \sqrt{10^2 + 8^2} = 12.81$, and $|z_2| = \sqrt{11^2 + (-6)^2} = 12.53$. Hence z_2 is closest to the origin.

For amount of rotation we have $\arg(z_1) = \tan^{-1}(8/10) = 0.675$ radians, and $\arg(z_2) = \tan^{-1}(11/-6) = -1.071$. Hence z_1 has rotated the least amount in absolute terms.

To find which complex number is closest to $z_3 = 1 + i$ we need to find the distance between z_1 and z_3 and compare this with the distance between z_2 and z_3. Hence

$$z_1 - z_3 = 10 + 8i - (1 + i) = 9 + 7i, \text{ implying } |z_1 - z_3| = \sqrt{9^2 + 7^2} = \sqrt{130}\,.$$

and

$$z_2 - z_3 = 11 - 6i - (1 + i) = 10 - 7i, \text{ implying } |z_2 - z_3| = \sqrt{10^2 + (-6)^2} = \sqrt{136}\,.$$

So z_1 is closest to z_3.

2) Notice that $z = 1 - \cos\theta - i\sin\theta$ is not in standard polar form. To express this equation in standard polar form we proceeed as follows: remember that by standard trig identities we have $\sin 2\theta = 2\sin\theta\cos\theta$, hence $\sin\theta = 2\sin\theta/2\cos\theta/2$. Also, $\cos 2\theta = 1 - 2\sin^2\theta$. Hence $\cos\theta = 1 - 2\sin^2\theta/2$. Therefore *z* becomes

$$z = 2\sin^2\frac{\theta}{2} - 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,.$$

Hence

$$z = 2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2} - i\cos\frac{\theta}{2}\right).$$

There are now two ways to proceed. We can either i) factorise *i* from this equation to obtain

$$z = -2i\sin\frac{\theta}{2}\left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right),$$

or ii) we can say

$$z = 2\sin\frac{\theta}{2}(\cos\alpha + i\sin\alpha)$$

where α is chosen so that $\cos\alpha = \sin\theta/2$ and $\sin\alpha = -\cos\theta/2$. Solving these two equation gives us $\alpha = (\theta - \pi)/2$.

Note that this version of the polar form is valid only if $\sin\theta/2$ is positive. If $\sin\theta/2$ is negative then we write

$$z = -2\sin\frac{\theta}{2}(-\cos\alpha - i\sin\alpha) = -2\sin\frac{\theta}{2}(\cos(\alpha - \pi) + i\sin(\alpha - \pi)) \,.$$

Two general comments are now in order about a complex number z. Firstly we now have two ways of expressing the coordinates of a complex number z in a plane:

the Cartesian form $z = x + iy$, and the polar form $z = r(\cos\theta + i\sin\theta)$.

There is also a third way of expressing the location of a point in the complex plane with the help of the complex conjugate. This can be seen as follows: since

$$z = x + iy$$

we have

$$z^* = x - iy \,.$$

Solving for x by adding gives

$$x = \frac{1}{2}(z + z^*)$$

and solving for y by subtracting gives

$$y = \frac{1}{2i}(z - z^*) \,.$$

We can therefore use (z, z^*) as coordinates to specify any complex number in the Argand diagram.

Secondly, note that the general form of the modulus of the polar form of z is

$$|z| = \sqrt{(r\cos\theta)^2 + (r\sin\theta)^2} = \sqrt{r^2(\cos^2\theta + \sin^2\theta)} = r \,,$$

as expected, and the general form of the argument of z is

$$\arg(z) = \tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta}\right) = \tan^{-1}(\tan\theta) = \theta \,,$$

as expected.

1.11.2 *Proof of the quadratic formula using polar form of a complex number*

This section is taken from "The Quadratic Formula Revisited Again", Peter A. Lindstrom, *Pi Mu Epsilon Journal*, Vol. 10, No. 6, (SPRING 1997), pp. 461-463.

It is possible to derive the quadratic formula for a quadratic $y = ax^2 + bx + c = 0$ when the complex roots x are in polar form. As such, let $x = r(\cos\theta + i\sin\theta)$, where $r > 0$ and $i^2 = -1$. If x is a solution to the aforementioned quadratic then

$$y = ar^2(\cos\theta + i\sin\theta)^2 + br(\cos\theta + i\sin\theta) + c = 0\,.$$

Our aim is to solve for $r\cos\theta$ and $r\sin\theta$. Hence (using $\cos^2\theta + \sin^2\theta = 1$ where appropriate), we have

$$[2a(r\cos\theta)^2 - ar^2 + br\cos\theta + c] + ir\sin\theta\,[2ar\cos\theta + b] = 0\,.$$

Equating *Re* and *Im* parts we have, respectively,

$$2a(r\cos\theta)^2 - ar^2 + br\cos\theta + c = 0 \tag{31}$$

and

$$r\sin\theta\,[2ar\cos\theta + b] = 0\,. \tag{32}$$

From (32) we have two solutions: $r\sin\theta = 0$ and $2ar\cos\theta + b = 0$. For the latter we obtain

$$r\cos\theta = -\frac{b}{2a}\,. \tag{33}$$

Substituting this into (31) we obtain

$$2a\left(-\frac{b}{2a}\right)^2 - ar^2 + b\left(-\frac{b}{2a}\right) + c = 0\,,$$

which simplifies to $r^2 = c/a$. Now using $(r\cos\theta)^2 + (r\sin\theta)^2 = r^2$ we have

$$\begin{aligned}
(r\sin\theta)^2 &= r^2 - (r\cos\theta)^2\,,\\
&= r^2 - \left(-\frac{b}{2a}\right)^2,\\
&= \frac{c}{a} - \left(-\frac{b}{2a}\right)^2,\\
&= \frac{4ac - b^2}{4a^2}\,.
\end{aligned}$$

Hence

$$r\sin\theta = \pm\frac{\sqrt{4ac - b^2}}{2a}\,. \tag{34}$$

Therefore, $x = r(\cos\theta + i\sin\theta)$ becomes

$$x = -\frac{b}{2a} \pm i\frac{\sqrt{4ac - b^2}}{2a}. \tag{35}$$

If $4ac - b^2 > 0$, equation (35) expresses the fact that the roots x of $y = f(x)$ are complex. If $4ac - b^2 < 0$, equation (35) can be transformed as follows:

$$\begin{aligned} x &= -\frac{b}{2a} \pm \sqrt{-1} \times \frac{\sqrt{4ac - b^2}}{2a}, \\ &= -\frac{b}{2a} \pm \frac{\sqrt{(-1)(4ac - b^2)}}{2a}. \end{aligned}$$

In other words,

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}, \tag{36}$$

which expresses the fact that the roots x of $y = f(x)$ are real.

Recall that one of the solutions to equation (32) is $r\sin\theta = 0$. Since $r > 0$ we have $\sin\theta = 0$. This means that the polar form of the roots are $x = r\cos\theta$, implying these roots are purely real.

1.11.3 Choosing the correct argument for z

Let us return to studying the polar form of a complex number via the following two examples:

i) if $z = -\sqrt{2} + i$ then $r = |z| = \sqrt{\left(-\sqrt{2}\right)^2 + 1^2} = \sqrt{3}$, and $\theta = \arg(z) = \tan^{-1} -1/\sqrt{2} \approx -0.62$ radians. Hence $z = \sqrt{3}(\cos(-0.62) + i\sin(-0.62)) = \sqrt{3}(\cos 0.62 - i\sin 0.62)$.

ii) if $z = -1 - i\sqrt{3}/2$ then $r = |z| = \sqrt{(-1)^2 + \left(-\sqrt{3}/2\right)^2} = \sqrt{7}/2$ and $\theta = \arg(z) = \tan^{-1} -\left(\sqrt{3}\right)/2/(-1) = 5\pi/22$. Hence $z = \sqrt{7}/2(\cos(5\pi/22) + i\sin(5\pi/22))$.

The answers $z = \sqrt{3}(\cos 0.62 - i\sin 0.62)$ and $z = \sqrt{7}/2(\cos(5\pi/22) + i\sin(5\pi/22))$, although numerically correct, do not correctly represent the original complex numbers. In fact, there are now two major issues we need to deal with, one dealing with the incorrect representation just mentioned, and the other dealing with the periodicity of sin and cos.

In terms of the incorrect representation of the two polar form answers in i) and ii) above we see that

- for i), $z = -\sqrt{2} + i$ lies in quadrant II of the Argand diagram, shown as z_1 in diagram (a) below. But in polar form $z = \sqrt{3}(\cos 0.62 - i\sin 0.62)$ is located in quadrant IV, shown as z_2 in the diagram (a) below.
- for ii), $z = -1 - i\sqrt{3}/2$ lies in quadrant III of the Argand diagram, shown as z_3 in diagram (b) below, whereas in polar form $z = \sqrt{7}/2(\cos(5\pi/22) + i\sin(5\pi/22))$ is located in quadrant I, shown as z_4 in the diagram (b) below.

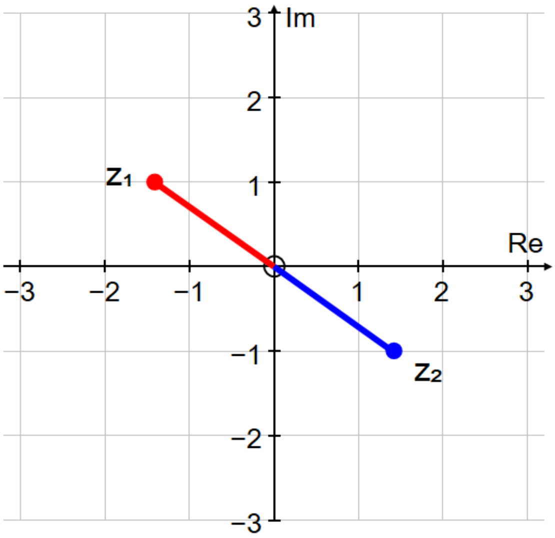


diagram (a)

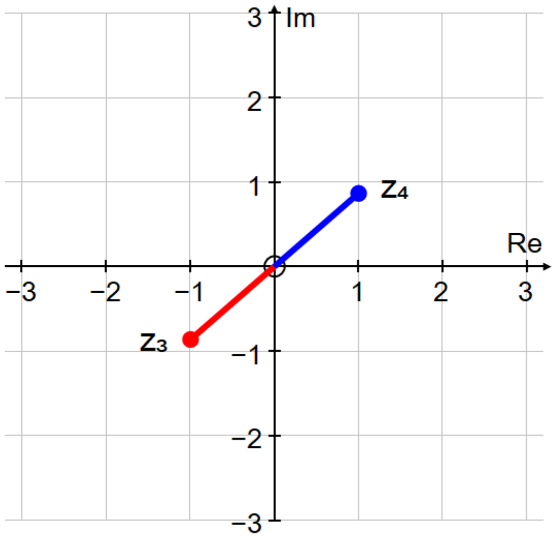
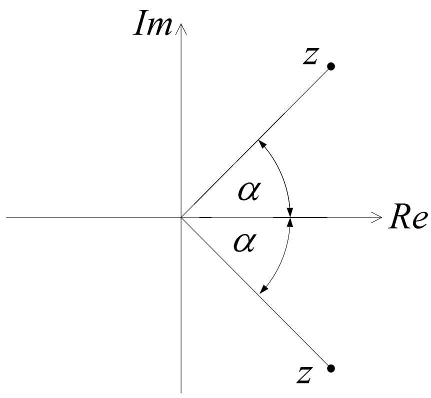


diagram (b)

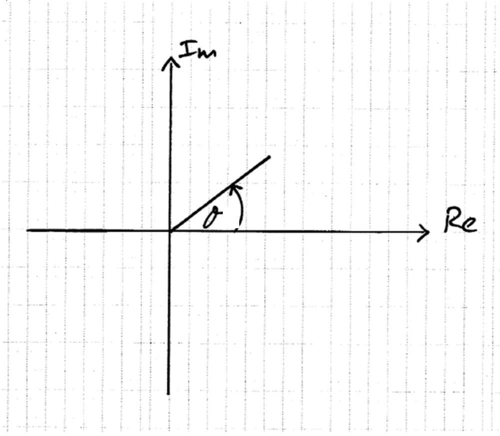
So what went wrong? In one sense, nothing went wrong. Our maths was correct in-and-of itself, so why did such a problem occur, and how do we correct it? Well, the reason why the error occurs is because of the nature of arctan, more precisely the interval over which arctan is defined.

For a complex number $z = a + ib$, we evaluate its argument as $\alpha = \tan^{-1} b/a$. Recall that the principle argument of α here is $-\pi/2 < \alpha < \pi/2$, as illustrated diagrammatically on the right.

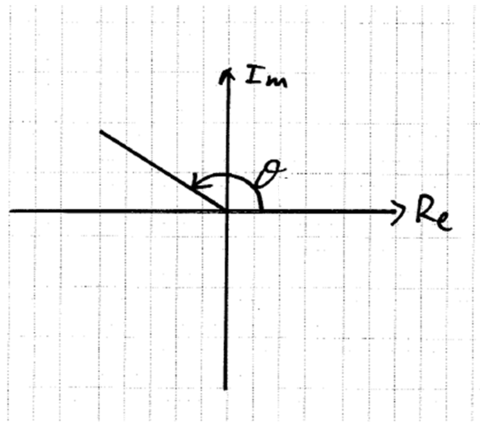


However, any complex number can lie in one of four possible locations, as shown in the diagrams below.

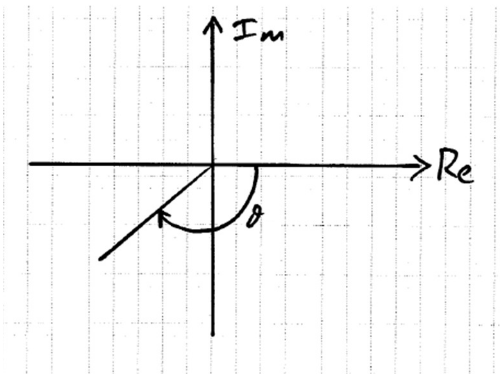
(a)



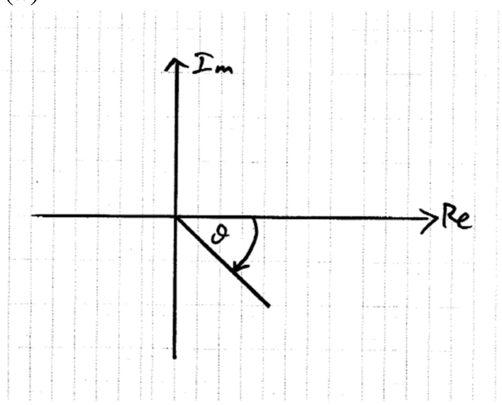
(*b*)



(*c*)



(*d*)

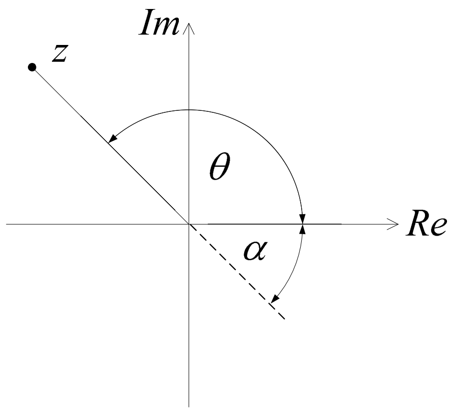


In each case we want to measure angle θ, the angle the complex number makes with the *Re* axis, but in evaluating arctan we will not always get θ. So what we have do is to make a relevant correction to α in order to account for the actual position of each complex number z. we do this by appropriately adding or subtracting π from α in order to obtain θ.

<u>*For diagrams (a) and (d)*</u>

Here a > 0. In this case α, calculated as $\tan^{-1} b/a$ and lying in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, already matches θ so no correction needs to be made to α. Hence $\theta = \tan^{-1} b/a$.

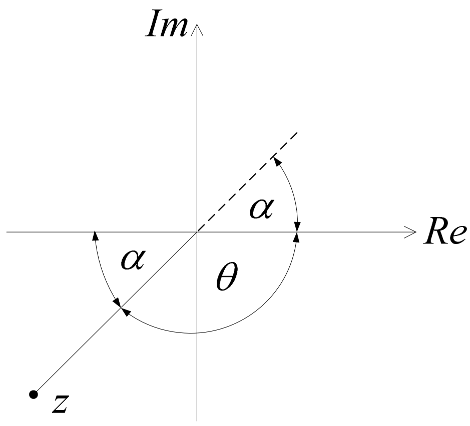
For diagram (b)



Here a < 0 and b > 0. In this case α, calculated as $\tan^{-1} b/a$ and lying in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, does not match θ so we need to correct this to

$$\theta = \pi + \alpha = \pi + \tan^{-1} b/a$$

For diagram (c)



Here a < 0 and b < 0. In this case α, calculated as $\tan^{-1} b/a$ and lying in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, does not match θ so we need to correct this to

$$\theta = \alpha - \pi = \tan^{-1} b/a - \pi$$

Summary

In summary we have the argument θ of a complex number $z = a + ib$ to be

$$\operatorname{Arg} z = \theta = \begin{cases} \tan^{-1}\dfrac{b}{a} & \text{for } a > 0 \\ \tan^{-1}\dfrac{b}{a} + \pi & \text{for } a < 0, b > 0 \\ \tan^{-1}\dfrac{b}{a} - \pi & \text{for } a < 0, b < 0 \end{cases}$$

along with

$$\operatorname{Arg} z = \theta = \begin{cases} 0 & \text{for } x \neq 0, y = 0 \\ \pi/2 & \text{for } x = 0, y > 0 \\ -\pi/2 & \text{for } x = 0, y < 0 \\ \text{undefined} & \text{for } x = 0, y = 0 \end{cases}$$

So the answers to example i) and ii) can now be corrected as follows:

- if $z = -\sqrt{2} + i$ then $r = |z| = \sqrt{\left(-\sqrt{2}\right)^2 + 1^2} = \sqrt{3}$. Since $a < 0$ we have $\theta = \arg(z) = \tan^{-1} -1/\sqrt{2} + \pi \approx 2.53$ radians. Hence $z = \sqrt{3}(\cos 2.53 + i \sin 2.53)$, which now lies in quadrant II;

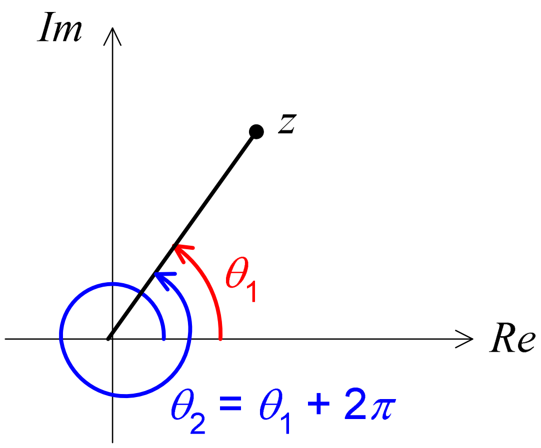
- if $z = -1 - i\sqrt{3}/2$ then $r = |z| = \sqrt{(-1)^2 + \left(-\sqrt{3}/2\right)^2} = \sqrt{7}/2$. Since $a < 0$ and $b < 0$ we have $\theta = \arg(z) = \tan^{-1} -\left(\sqrt{3}\right)/2/(-1) - \pi = -17\pi/22$. Hence

  $z = \sqrt{7}/2(\cos(-17\pi/22) + i \sin(-17\pi/22)) = \sqrt{7}/2(\cos(17\pi/22) - i \sin(17\pi/22))$,

  which now lies in quadrant III.

The polar form of the complex numbers now correctly represents the original Cartesian forms.

The second issue is that, because of the periodic nature of sin and cos, we can represent any complex number $z = a + ib$ in polar form in an infinite number of ways. In example 1) above we transformed $z = 1 + i$ into $z = \sqrt{2}(\cos \pi/4 + i \sin \pi/4)$. However, this can also be expressed as $z = \sqrt{2}(\cos 9\pi/4 + i \sin 9\pi/4)$ or $z = \sqrt{2}(\cos 17\pi/4 + i \sin 17\pi/4)$. In fact, any complex number $z = \sqrt{2}(\cos(\pi/4 \pm 2k\pi) + i \sin(\pi/4 \pm 2k\pi))$, where $k = 0, 1, 2, \ldots$, express the samne Cartesian form $z = 1 + i$. The same is true for any complex number in polar form.



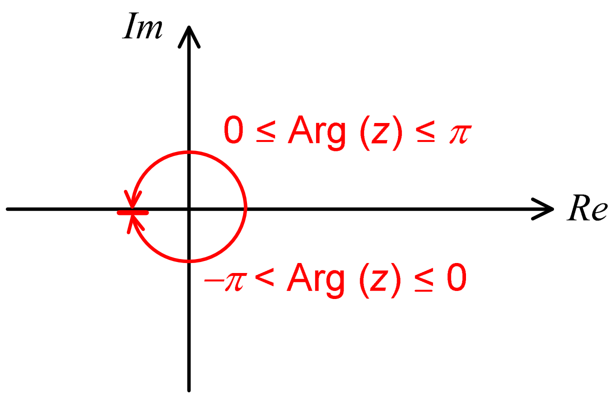
More than one choice for the argument θ

The question now is, Which polar form do we choose? Well, we need to decide on what we will take as the first argument, or *principal argument*, of our complex number. To do this we need to define a principal interval over which we measure θ. Since sin and cos have period 2π we will choose our interval to be 2π long. For example, we could choose $[0, 2\pi)$ or $(-\pi, \pi]$. The interval we choose to measure θ over is $(-\pi, \pi]$. This is chosen partly as a convention and partly for technical reasons to do with more advanced complex number work.

Hence, from now on, the principal argument of z, denoted $\text{Arg}(z)$, is given by

$$-\pi < \text{Arg}(z) \leq \pi\,. \tag{37}$$

Note that some books define the principal angle to be $0 \leq \text{Arg}(z) < 2\pi$. This is ok. The aim is to define the principal argument so as to include 0 in its interval.



The interval of Arg(z): Angles cannot go beyond π, and negative angles cannot equal −π, as identified in the diagram by the thick red line.

So, for the following examples we have

- for $z = 1 + i$ the principal argument is $\theta = \text{Arg}(z) = \pi/4$, with other arguments being $\pi/4 \pm 2n\pi$. Hence $z_1 = \sqrt{2}(\cos \pi/4 + i \sin \pi/4)$, with $z_2 = \sqrt{2}(\cos 9\pi/4 + i \sin 9\pi/4)$, $z_3 = \sqrt{2}(\cos 17\pi/4 + i \sin 17\pi/4)$, etc.

- for $z = \sqrt{3} - i$ the principal argument is $\theta = \text{Arg}(z) = \tan^{-1} -1/\sqrt{3} = -\pi/6$, with other arguments being $-\pi/6 \pm 2n\pi$. Hence $z_1 = 2(\cos(-\pi/6) + i \sin(-\pi/6)) = 2(\cos \pi/6 - i \sin \pi/6)$, with $z_2 = 2(\cos(5\pi/6) + i \sin(5\pi/6))$, etc.

- for $z = -\sqrt{2} + i$ the principal argument is $\theta = \text{Arg}(z) = \tan^{-1} -1/\sqrt{2} + \pi \approx 2.53$ radians. Hence $z_1 = \sqrt{3}(\cos 2.53 + i \sin 2.53)$, with other complex numbers following at 2π intervals, i.e. $z_2 = \sqrt{3}(\cos 5.76 + i \sin 5.76)$, etc.

- for $z = -1 - i\sqrt{3}/2$ the principal argument is $\theta = \text{Arg}(z) = \tan^{-1} -(\sqrt{3})/2/(-1) - \pi = -17\pi/22$. Hence $z_1 = \sqrt{7}/2(\cos(17\pi/22) - i\sin(17\pi/22))$, with other complex numbers following at 2π intervals, i.e. $z_2 = \sqrt{7}/2(\cos(5\pi/22) + i\sin(5\pi/22))$, etc.

In general we therefore have

$$\arg(z) = \text{Arg}(z) \pm 2k\pi. \tag{38}$$

for $k = 0, 1, 2, 3$ ... Therefore, given $\arg(z)$ for which the principal argument is $\text{Arg}(z) = \theta$ then $\arg(z)$ consists of the set of values

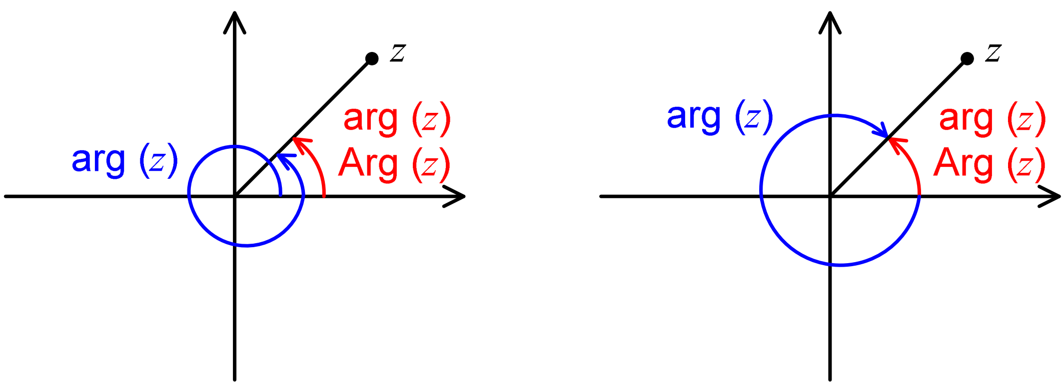
$$\arg(z) = \{\dots, \theta - 4\pi, \theta - 2\pi, \theta, \theta + 2\pi, \theta + 4\pi, \dots\}.$$

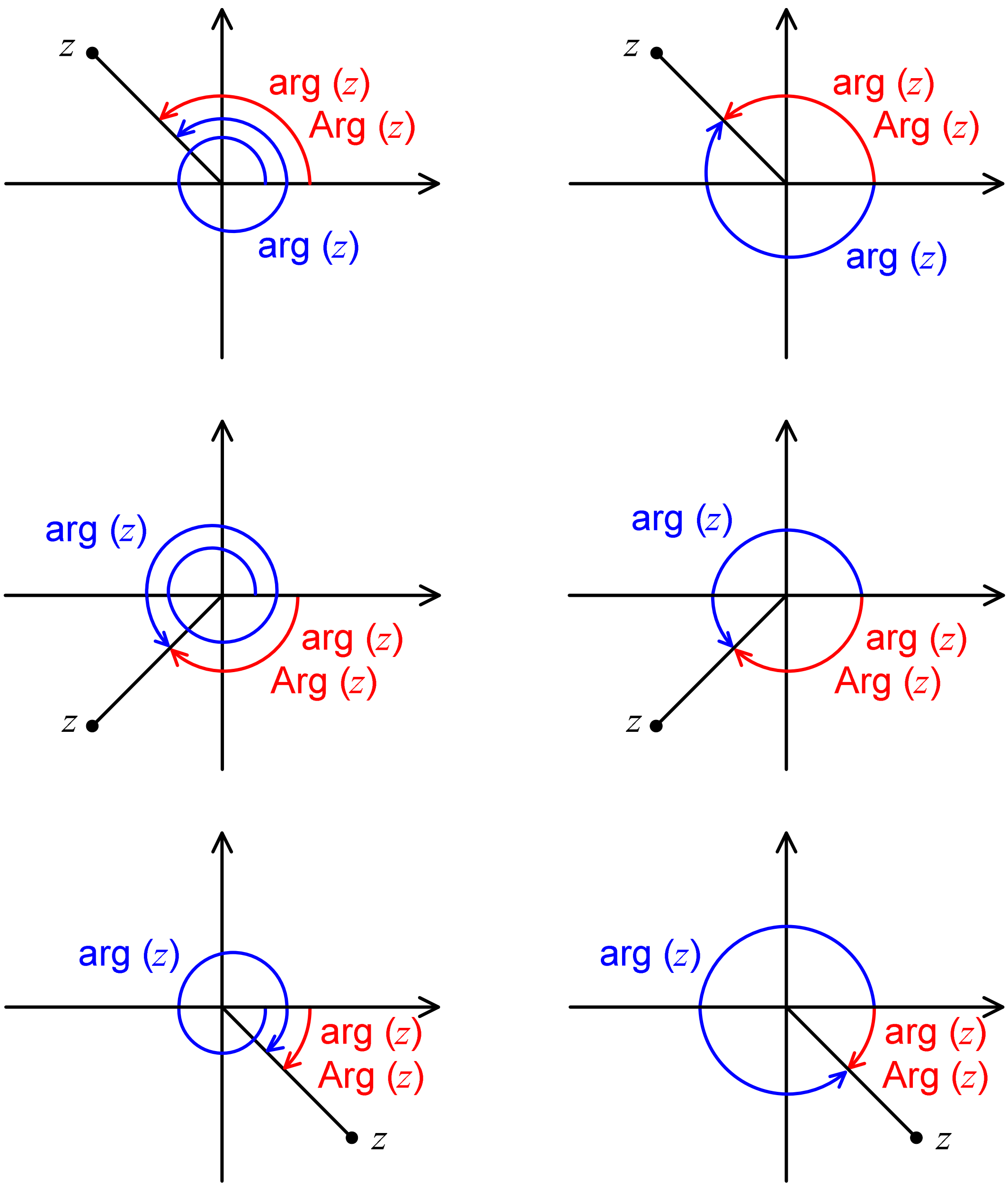
Therefore that $\arg(z)$ represents any angle of any size, for example, choosing three arguments at random, $\arg(z_1) = 3\pi/4$ or $\arg(z_2) = 5\pi/4$ or $\arg(z_3) = 9\pi/4$. However, $\text{Arg}(z)$ only ever represent an angle in the interval $-\pi < \text{Arg}(z) \le \pi$. Hence, $\text{Arg}(z_1) = 3\pi/4$, $\text{Arg}(z_2) = -3\pi/4$, and $\text{Arg}(z_3) = \pi/4$.

The following four examples illustrate this further:

- for $z = 1 + i$: $\text{Arg}(z) = \frac{\pi}{4}$, whereas $\arg(z) = \cdots, -\frac{17\pi}{4}, -\frac{9\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{9\pi}{4}, \frac{17\pi}{4}, \dots$
- for $z = \sqrt{3} - i$: $\text{Arg}(z) = -\frac{\pi}{6}$, whereas $\arg(z) = \cdots, -\frac{25\pi}{6}, -\frac{13\pi}{6}, -\frac{\pi}{6}, \frac{11\pi}{6}, \frac{23\pi}{6}, \dots$
- for $z = -\sqrt{2} + i$: $\text{Arg}(z) = 2.53$, whereas $\arg(z) = \cdots - 10.03, -3.75, 2.53, 8.81, 15.10, \dots$
- for $z = -1 - i\sqrt{3}/2$: $\text{Arg}(z) = -2.43$, whereas $\arg(z) = \cdots, -8.71, -2.43, 3.85, 10.14, \dots$

A set of diagrams illustrating a representative $\arg(z)$ and $\text{Arg}(z)$ for complex numbers in each of the four quadrant is shown below:





Examples

Convert the following complex number into polar form, using only the principal argument:

i) $z = 5 - 5i$ ii) $z = -2 + 2\sqrt{3}.i$ iii) $z = -12 - 5i$

iv) $z = -10$ v) $z = 6i$ vi) $z = 3/(-1 + i)$

vii) $z = (4 + i)(1 - i)(2 + i)$ viii) $z = (1 + i)^5$

Solutions

i) For $z = 5 - 5i$ we have $r = |z| = \sqrt{5^2 + (-5)^2} = 5\sqrt{2}$. The argument $\theta = \arg(z) = \operatorname{Arg}(z) = \tan^{-1}(-5/5) = -\pi/4$.

Hence

$$z = 5\sqrt{2}(\cos(-\pi/4) + i\sin(-\pi/4)) = 5\sqrt{2}(\cos(\pi/4) - i\sin(\pi/4))\,.$$

ii) For $z = -2 + 2\sqrt{3}.i$ we have $r = |z| = \sqrt{(-2)^2 + (2\sqrt{3})^2} = 4$. The argument $\theta = \arg(z) = \mathrm{Arg}(z) + \pi = \tan^{-1}\left(-2/(2\sqrt{3})\right) + \pi = -\pi/6 + \pi = 5\pi/6$. Hence

$$z = 4(\cos(5\pi/6) + i\sin(5\pi/6))\,.$$

iii) For $z = -12 - 5i$ we have $r = |z| = \sqrt{(-12)^2 + (-5)^2} = 13$. The argument $\theta = \arg(z) = \mathrm{Arg}(z) - \pi = \tan^{-1}(-5/(-12)) - \pi = -2.75$ radians. Hence

$$z = 13(\cos(-2.75) + i\sin(-2.75)) = 13(\cos(2.75) - i\sin(2.75))\,.$$

iv) For $z = -10$ we have $r = |z| = \sqrt{(-10)^2 + 0^2} = 10$. The argument $\theta = \arg(z) = \mathrm{Arg}(z) + \pi = \tan^{-1}(0/10) + \pi = \pi$. Hence

$$z = 10(\cos\pi + \sin\pi)\,.$$

v) For $z = 6i$ we have $r = |z| = \sqrt{0^2 + 6^2} = 6$. The argument $\theta = \arg(z) = \mathrm{Arg}(z) = \tan^{-1}(6/0) = \pi/2$. Hence

$$z = 6\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right).$$

vi) For $z = 3/(-1+i)$ we have

$$z = \frac{3}{-1+i}.\frac{-1+i}{-1+i} = \frac{1}{2}(-3-3i)\,.$$

Hence $r = |z| = \sqrt{(-3/2)^2 + (-3/2)^2} = 3/\sqrt{2}$. The argument $\theta = \arg(z) = \mathrm{Arg}(z) - \pi = \tan^{-1}((-3/2)/(-3/2)) - \pi = -3\pi/4$. Hence

$$z = \frac{3}{\sqrt{2}}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right) = \frac{3}{\sqrt{2}}\left(\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4}\right).$$

vii) For $z = (4+i)(1-i)(2+i)$ we expand all three factors to obtain $z = 13 - i$. Hence $r = |z| = \sqrt{13^2 + (-1)^2} = \sqrt{170}$. The argument $\theta = \arg(z) = \mathrm{Arg}(z) = \tan^{-1}(-1/13) = -0.077$. Hence

$$z = \sqrt{170}(\cos(-0.077) + i\sin(-0.077)) = \sqrt{170}(\cos(0.077) - i\sin(0.077))\,.$$

viii) For $z = (1+i)^5$ we obtain $z = 1 + 5i + 10i^2 + 10i^3 + 5i^4 + i^5 = -4 - 4i$ using Pascal's triangle. Hence $r = |z| = \sqrt{(-4)^2 + (-4)^2} = 2\sqrt{8}$. The argument $\theta = \arg(z) = \operatorname{Arg}(z) - \pi = \tan^{-1}(-4/-4) - \pi = -3\pi/4$. Hence

$$z = 2\sqrt{8}(\cos(-3\pi/4) + i\sin(-3\pi/4)) = 2\sqrt{8}(\cos(3\pi/4) - i\sin(3\pi/4))\,.$$

Notice that these last three examples were more or less laborious to do since we first had to do some simplifying arithmetic before we could convert the complex numbers into polar form. in section 0 we will see a much simpler way of being able to perform multiplication and division).

More examples

1) Write the following complex numbers in terms of their principal arguents, then write them in the form $x + iy$:

   i) $z = 5(\cos(7\pi/6) + i\sin(7\pi/6))$ ii) $z = 8\sqrt{2}(\cos(11\pi/4) + i\sin(11\pi/4))$

   iii) $z = 6(\cos(\pi/8) - i\sin(\pi/8))$

   Solutions

   Remember that the principal argument is given by $-\pi < \theta \leq \pi$ hence

   i) $z = 5(\cos(7\pi/6) + i\sin(7\pi/6)) = 5(\cos(\pi/6) + i\sin(\pi/6))$. Hence in Cartesian form we have

   $$z = \frac{5\sqrt{3}}{2} + \frac{5}{2}i\,.$$

   ii) $z = 8\sqrt{2}(\cos(11\pi/4) + i\sin(11\pi/4)) = 8\sqrt{2}(\cos(3\pi/4) + i\sin(3\pi/4))$. Hence in Cartesian form we have

   $$z = 8\sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = -8 + 8i\,.$$

   iii) $z = 6(\cos(\pi/8) - i\sin(\pi/8))$ is already in principal argument form. However, converting into standard form we have $z = 6(\cos(-\pi/8) + i\sin(-\pi/8))$ which in Cartesian form is $z = 5.54 - 2.30i$.

2) Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. If $z_1 = z_2$ how are r_1 and r_2 related? How are θ_1 and θ_2 related?

<u>*Solution:*</u> For the moduli we have $|z_1| = \sqrt{(r_1\cos\theta_1)^2 + (r_1\sin\theta_1)^2} = \sqrt{(r_1)^2(1)} = r_1$. Similarly $|z_2| = \sqrt{(r_2\cos\theta_2)^2 + (r_2\sin\theta_2)^2} = \sqrt{(r_2)^2(1)} = r_2$. Hence if $z_1 = z_2$ then $r_1 = r_2$.

For the argument we have $\arg(z_1) = \theta_1$. Hence, if $z_1 = z_2$, $\arg(z_2) = \theta_2 = \theta_1$. But by the periodicity of sin and cos, $\arg(z_2)$ is also equal to $\theta_2 = \theta_1 \pm 2n\pi$. Hence $\arg(z_2)$ is any 2π multiple of $\arg(z_1)$.

3) One person believes that $\arg(z^*) = -\arg(z)$. For example, if $z = 1 + i$ then $z^* = 1 - i$. Here $\arg(z) = \pi/4$ and $\arg(z^*) = -\pi/4$. Another person disagrees with this, and claims to have a counter-example: if $z = i$ then $z^* = -i$. Hence $\arg(z) = \pi/2$ and $\arg(z^*) = 3\pi/2$ implying $\arg(z^*) \neq -\arg(z)$. Who is correct and why?

<u>*Solution*</u>

Remember that $\arg(z) = \text{Arg}(z) \pm 2n\pi$, where $n = 0,1,2,3,\ldots$, and that $-\pi < \text{Arg}(z) \leq \pi$. So for $z = 1 + i$, $\arg(z) = \text{Arg}(z) = \pi/4$, and for $z^* = 1 - i$, $\arg(z) = \text{Arg}(z) = -\pi/4$. Hence, $\arg(z^*) = -\arg(z)$ in this case.

For $z = i$, $\arg(z) = \text{Arg}(z) = \pi/2$, and for $z = -i$, $\arg(z) = \text{Arg}(z) = -\pi/2$, i.e. $\pi/2$ measured in the clockwise direction, not $3\pi/2$ which is a measurement in the anti-clockwise direction. Hence, $\arg(z^*) = -\arg(z)$ in this case also.

4) Suppose we have a complex number $z = 1 + \cos\alpha + i\sin\alpha$, where $-\pi < \theta \leq \pi$. This is not in the standard polar form $z = r(\cos\theta + i\sin\theta)$. So, in order to transform this into standard polar form we find r and θ as follows:

   1) $r = |z| = \sqrt{(1+\cos\alpha)^2 + \sin^2\alpha} = \sqrt{2(1+\cos\alpha)} = \sqrt{4\cos^2(\alpha/2)} = 2|\cos(\alpha/2)|$ where, since we take r as positive, the modulus sign indicates that we want the magnitude of cos;

   2) we find the argument as follows: the general argument θ is the angle made by the line/length of z with the *Re* axis. By our previous calculation the length of the line is given by $2|\cos(\alpha/2)|$, hence the angle z makes with the *Re* axis is $\alpha/2$.

      - if $\alpha \in (-\pi, 0)$ then $\alpha/2 \in (-\pi/2, 0)$ implying that z is in the fourth quadrant. Hence

$$\theta = \tan^{-1}\left(\frac{\sin\alpha}{1+\cos\alpha}\right) = \tan^{-1}\left(\tan\frac{\alpha}{2}\right) = \frac{\alpha}{2}.$$

This therefore gives us

$$z = (2\cos(\alpha/2\ ))(1 + \cos(\alpha/2) + i\sin(\alpha/2)).$$

- if $\alpha = 0$ then

$$z = 2(1 + 1) = 4\,.$$

- if $\alpha \in (0, \pi)$ then $\alpha/2 \in (0, \pi/2)$ implying that z is in the first quadrant. Hence

$$\theta = \tan^{-1}\left(\frac{\sin\alpha}{1 + \cos\alpha}\right) = \tan^{-1}\left(\tan\frac{\alpha}{2}\right) = \frac{\alpha}{2}\,.$$

  This therefore gives us

$$z = 2\cos(\alpha/2\ )\,(1 + \cos(\alpha/2) + i\sin(\alpha/2)).$$

- if $\alpha = \pi$ then $z = 0$.

Exercises:

1) Suppose z_1 is located in the first quadrant. For each z_2 below state, with reasons, the quadrant in which $z_1 z_2$ is located:

   i) $z_2 = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ ii) $z_2 = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$

   iii) $z_2 = -i$ iv) $z_2 = -1$

1.11.4 The geometric effect of powers of i

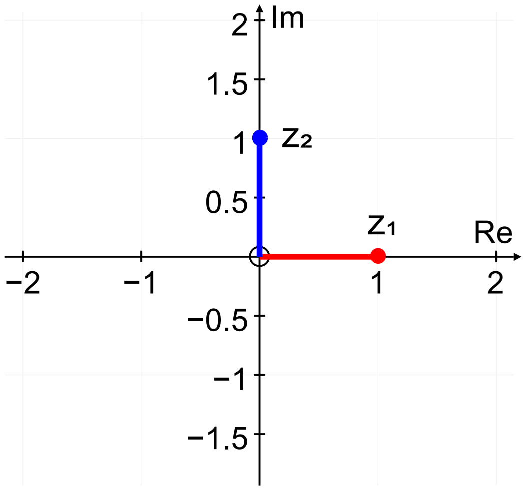
In section 1.7.2 we saw the geomertic effect of multiplying by *i*. Here we will look at this effect agin, but this time from the perspective of the polar representation of *i*. So, let us take $z_1 = 1$. In polar form this becomes

$$z_1 = \cos 0 + i\sin 0\,.$$

If we now consider $z_2 = i.z_1 = i$, it polar form is

$$z_2 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\,.$$

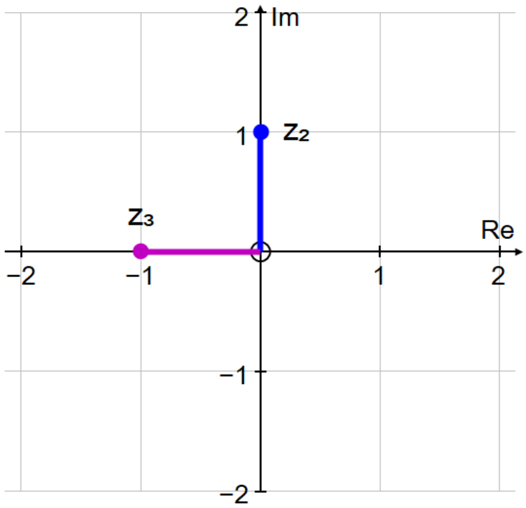
We can now see much more obviously that the effect of multplying by *i* is of rotate z_1 by $\pi/2$ radians, as illustrated below.



Multiplying z_2 by i gives $z_3 = i.z_2 = i^2 z_1 = -1$, which in polar form is given by

$$z_3 = \cos \pi + i \sin \pi \, .$$

Again we see that the effect of multplying by i is of rotate z_2 by $\pi/2$ radians, as illustrated below.



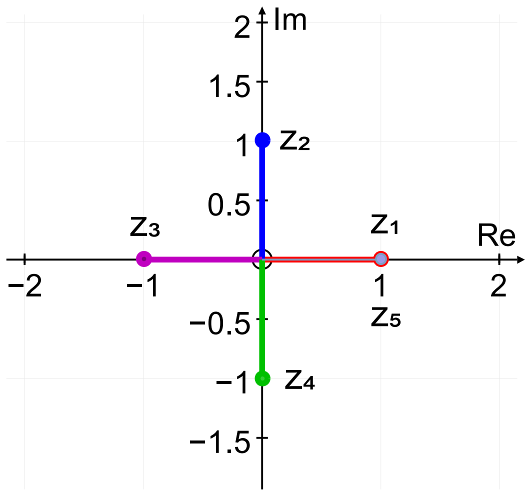
Multiplying z_3 by i gives $z_4 = i.z_3 = i^3 z_1 = -i$ and multiplying z_4 by i gives $z_5 = i.z_4 = i^4 z_1 = 1$. In polar form these are given by $z_4 = \cos(-\pi/2) + i \sin(-\pi/2)$ and $z_5 = \cos 0 + i \sin 0$. Note that although we don't do this, z_4 and z_5 can also be written as

$$z_4 = \cos\left(\frac{3\pi}{2}\right) + i \sin\left(\frac{3\pi}{2}\right)$$

and

$$z_5 = \cos 2\pi + i \sin 2\pi \, .$$

From these we can see that the effect of continually multiplying by i is to continually rotate by $\pi/2$ radians. The sequence of complex numbers z_1, z_2, z_3, z_4, and z_5 is illustrated below.



If we continue to multiply by i we end up with

$$z_6 = i.z_5 = \cos\left(\frac{5\pi}{2}\right) + i\sin\left(\frac{5\pi}{2}\right) = i\,,$$

$$z_7 = i.z_6 = \cos\left(\frac{6\pi}{2}\right) + i\sin\left(\frac{6\pi}{2}\right) = -1\,,$$

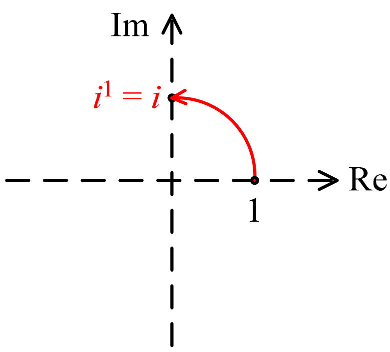
etc. The rotation effect of multiplying by i can then be summarised as follows:

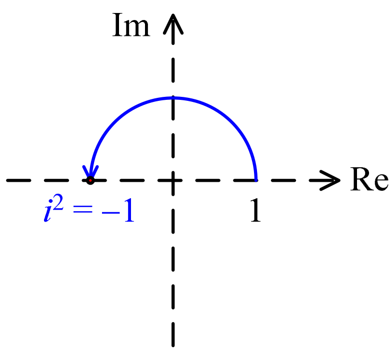
n:	0	4	8	12	...	gives	$i^n = 1$
n:	1	5	9	13	...	gives	$i^n = i$
n:	2	6	10	14	...	gives	$i^n = -1$
n:	3	7	11	15	...	gives	$i^n = -i$

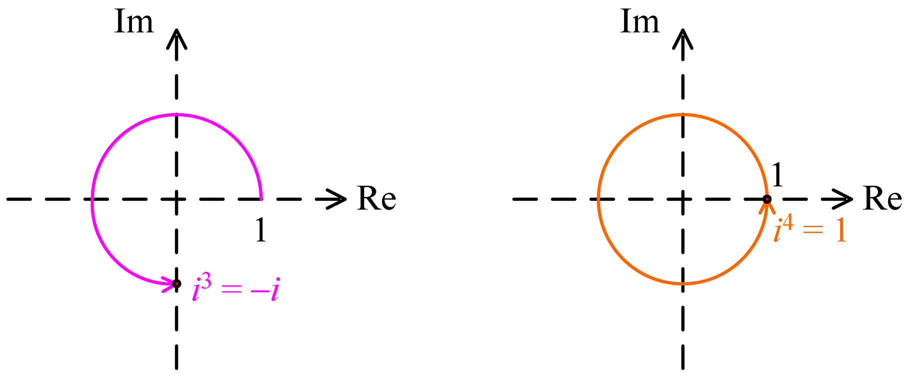
which can all be condensed into the following formula:

$$i^{4n} = 1\,, \qquad i^{4n+1} = i\,, \qquad i^{4n+2} = -1\,, \qquad i^{4n+3} = -i\,,$$

for $n = 0, 1, 2, 3, 4$ ... The cases for $n = 0, 1, 2, 3, 4$ are illustrated below:







The rotation effect of i: $1 \times i^n$ *for n = 0, 1, 2, 3, 4.*

In general, when looking to simplify expressions involving powers of i we tend to look for multiples of i^2 or i^4. For example,

$$i^{33} = (i^4)^8.i = i\,, \qquad i^{68} = (i^4)^{17} = 1\,, \qquad i^{87} = (i^4)^{21}.i^3 = i^3\,, \qquad i^{102} = (i^4)^{25}.i^2 = -1\,.$$

All of the above illustrates the effect of multiplying by a number *i*. However, it is also possible to see i as an operator, i.e. something which has an effect on an object which results in that object being tranformed, just as +, –, ×, ÷, d/dx are operator. Louis Diamond describes this well. Considering a complex number as a "directed magnitude", and starting with a line segment OA along the real axis, he say:

> "Argand considered that the "multiplication" of *A* by i was the algebraic equivalent of a geometric counterclockwise rotation about O [the origin] of a directed line segment OA through an angle whose measure was $\pi/2$ radians. The terminal point of the rotated segment became Ai, the length of the segment being unaltered. In other words i acted upon the directed magnitude OA to change its direction by 90° counterclockwise without changing its magnitude. A second "multiplication" by i rotates the line segment Ai counterclocwise about O through $\pi/2$ radians so that its terminal point is now $-A$. These two succesive operations by i constitute the operation i^2 which is equivalent to the ordinary multiplication of A by -1. The operation i^3 brings the terminal point to $-Ai$, or $i^3 = i^2.i = -1i = -i$. The operation i^3 is equivalent to the operation $-i$. By defining the operation $-i$ as a clockwaise rotation about O through $\pi/2$ radians, the operation $i^3 = -i$ is consistent. Four successive operations by i, or the operation i^4, brings the terminal point of the line segment back to $+A$. This is equivalent to the

ordinary multiplication of $+A$ by $+1$. $i^2 i^2 = (-1)(-1) = +1$. Complex numbers are thus conceived of, not simply as magnitudes, but as directed magnitudes. Particularly note that i as an operator has no effect upon magnitude but only upon the direction of the magnitude.

[...] In a similar manner we can regard the factor $(\cos\phi + i\sin\phi)$ as an operator or direction coefficient which rotates any directed magnitude counterclockwise through an angle [phi] radians and which leaves the magnitude unchanged. The factor $R(\cos\phi + i\sin\phi)$ acts in exactly the same manner but multiplies the magnitude by R. The operator $1(\cos\pi/2 + i\sin\pi/2)$ completely fulfills the earlier definition of *i*.

[...] The operator $[\cos(-\phi) + i\sin(-\phi)]$ rotates a complex number through an angle $-\phi$. [...] [This is equivalent to] $(\cos\phi - i\sin\phi)$. If a complex number is operated upon by $(\cos\phi + i\sin\phi)$ and then by $(\cos\phi - i\sin\phi)$ the rotation through ϕ and then through $-\phi$ with magnitude unchanged leaves the complex number unchanged. The operator product $(\cos\phi + i\sin\phi)(\cos\phi - i\sin\phi) = \cos^2\phi + \sin^2\phi = 1$, i.e. it leaves magnitude and direction unaltered."

("Introduction to complex numbers", Louis E. Diamond, *Mathematics Magazine*, Vol 30, No. 5, (May - Jun., 1957), pp233-249)

1.12 On exponentiation of complex numbers: DeMoivre's theorem

Let us return to example viii) on p114. Here we wanted to find $z = (1+i)^5$. To do this we performed some simplifying arithmetic before being able to convert z into polar form. In this case we were lucky enough to be able to use Pascal's triangle to expand z easily and quickly. Now suppose we wanted to find $z = (0.3487 - 6.149i)^{23}$. Although this can be expanded using the binomial theorem it would be quite laborious.

However, there is a much quicker and more powerful way of performing the latter operation, this way being called *DeMoivre's theorem*. DeMoivre's theorem reduces the powering operation to a multiplication operation, thus allowing us to expand a binomial expression however large the power.

Expressing a complex number in polar form is very very useful for dealing with complex numbers. Amongst other things it allows us perform multiplication, division and taking roots much more easily, as we shall see.

1.12.1 DeMoivre's theorem

Consider, therefore, the following complex number.

$$z = r(\cos\theta + i\sin\theta)\ .$$

Squaring this number we have

$$\begin{aligned} z^2 &= r^2(\cos\theta + i\sin\theta)^2\ , \\ &= r^2(\cos^2\theta + 2i\sin\theta\cos\theta - \sin^2\theta)\ ; \\ &= r^2(\cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta)\ . \end{aligned}$$

Using standard trig identities on the *Re* and *Im* terms in the brackets we have

$$z^2 = r^2(\cos 2\theta + i.\sin 2\theta)\ .$$

Is it just a coincidence that the squaring of z leads simply to the squaring of the modulus and the doubling of the argument? What happens if we take the cube of z?

$$\begin{aligned} z^3 &= r^3(\cos\theta + i\sin\theta)^3\ , \\ &= r^3(\cos\theta + i\sin\theta)^2(\cos\theta + i\sin\theta)\ ; \\ &= r^3(\cos 2\theta + i\sin 2\theta)(\cos\theta + i\sin\theta)\ ; \\ &= r^3(\cos 2\theta + i\sin 2\theta)(\cos\theta + i\sin\theta)\ ; \\ &= r^3\big(\cos 2\theta\,.\cos\theta - \sin 2\theta\,.\sin\theta + i(\cos 2\theta\,.\sin\theta + \sin 2\theta\,.\cos\theta)\big)\ ; \\ &= r^3(\cos 3\theta + i\sin 3\theta)\ , \end{aligned}$$

again by use of trig identities. The above seems to suggest the following pattern

$$z^4 = r^4(\cos 4\theta + i\sin 4\theta)\ ,$$

$$z^5 = r^5(\cos 5\theta + i\sin 5\theta)\ ,$$

and in general it seems we have

$$z^n = r^n(\cos n\theta + i\sin n\theta)\ ,$$

at least for positive integer powers. If this is true we will have an incredibly simple way of powering complex number when the power is a positive integer. In fact, this last expression is true, not only for $n \in \mathbb{N}$ but also for $n \in \mathbb{R}$, and is know as *DeMoivre's theorem*.

1.12.2 Proof of DeMoivre's theorem (up to rational powers)

We will now prove that DeMoivre's theorem in stages, where the first stages is to show it is true for positive integer powers. We are therefore claiming that if $z = r(\cos\theta + i\sin\theta)$,

$$[r(\cos\theta + i.\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)\,,$$

is true for $n \in \mathbb{N}$.

Proof part 1: Let $P(n)$ be $[r(\cos\theta + i.\sin\theta)]^n = r^n(\cos n\theta + i.\sin n\theta)$

1. Base case: Let $n = 1$. Therefore the left hand side becomes

$$[r(\cos\theta + i.\sin\theta)]^1 = r(\cos\theta + i\sin\theta)\,,$$

and the right hand side becomes

$$r^1(\cos(1\times\theta) + i\sin(1\times\theta)) = r(\cos\theta + i\sin\theta)\,.$$

The left hand side equals the right hand side, hence $P(1)$ is true.

2. Inductive assumption – Let $n = k$: Let $P(k)$ be true for some positive integer k where $1 \le k \le n$. Then we have

$$P(k)\colon [r(\cos\theta + i.\sin\theta)]^k = r^k(\cos k\theta + i\sin k\theta)\,.$$

3. Let $n = k + 1$. We want to show that $P(k) \Longrightarrow P(k+1)$. Hence multiplying $P(k)$ by $r(\cos\theta + i\sin\theta)$ we obtain, by using our inductive assumption,

$$r^k(\cos k\theta + i\sin k\theta).r(\cos\theta + i\sin\theta) = r^{k+1}[(\cos k\theta\,.\cos\theta - \sin k\theta\,.\sin\theta)$$
$$\times\, i(\cos k\theta\,.\sin\theta - \sin k\theta\,.\cos\theta)]\,.$$

Using standard trig identities we end up with

$$r^k(\cos k\theta + i\sin k\theta).r(\cos\theta + i\sin\theta) = r^{k+1}(\cos(k+1)\theta + i\sin(k+1)\theta)\,,$$

which is $P(k+1)$, which is what we wanted to show. Hence $P(k) \Longrightarrow P(k+1)$, and since $P(1)$ is true we have $P(n)$ is true for all $n \in \mathbb{N}$.

■

For example, if $z = 1 + i$ then we can find z^2 by first converting to polar form and then using the above result. Hence, $r = |z| = \sqrt{2}$, and $\theta = \arg(z) = tan^{-1}\,1 = \pi/4$. Therefore,

$$z = \sqrt{2}(\cos\pi/4 + i\sin\pi/4)\,,$$

and

$$z^2 = \left(\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right)^2 = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^2 = 2\left(\cos\frac{\pi}{2} + i.\sin\frac{\pi}{2}\right) = 2i\,.$$

Furthermore, we can now evaluate any power of z with very little extra effort. For example, $z^4 = \left(\sqrt{2}\right)^4(\cos\pi/4 + i\sin\pi/4)^4 = 4(\cos(4\pi/4) + i\sin(4\pi/4)) = 4(\cos\pi + i\sin\pi) = -4$; or $z^{10} = \left(\sqrt{2}\right)^{10}(\cos\pi/4 + i\sin\pi/4)^{10} = \left(\sqrt{2}\right)^{10}(\cos(10\pi/4) + i\sin(10\pi/4)) = i\left(\sqrt{2}\right)^{10}$.

The question now is, Does this also work when n is negative? Yes. To prove this we set $n = -m$ where m is a positive integer. So we are claiming that if $z = r(\cos\theta + i\sin\theta)$,

$$[r(\cos\theta + i\sin\theta)]^{-m} = r^{-m}(\cos(-m\theta) + i\sin(-m\theta)),$$

is true.

Proof part 2: The left hand side of the above expression can be written as

$$[r(\cos\theta + i\sin\theta)]^{-m} = r^{-m}.\frac{1}{(\cos\theta + i\sin\theta)^m}.$$

By the previous proof we have

$$[r(\cos\theta + i\sin\theta)]^{-m} = r^{-m}.\frac{1}{\cos m\theta + i\sin m\theta}.$$

Multiplying by the conjugate of the denominator we have

$$[r(\cos\theta + i\sin\theta)]^{-m} = r^{-m}.\frac{1}{\cos m\theta + i\sin m\theta}.\frac{\cos m\theta - i\sin m\theta}{\cos m\theta - i\sin m\theta},$$

which ultimately simplfies to

$$[r(\cos\theta + i\sin\theta)]^{-m} = r^{-m}.\frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta} = r^{-m}(\cos(-m\theta) + i\sin(-m\theta)),$$

which is what we wanted to prove. So we now know that

$$[r(\cos\theta + i.\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

is true for all $n \in \mathbb{Z}$. ■

For example, if $z = -2 + i\sqrt{3}$ then we can find z^{-3} by first converting to polar form and then using the above result. Hence $r = |z| = \sqrt{7}$ and $\theta = \arg(z) = \tan^{-1}\left(-\sqrt{3}/2\right) + \pi \approx 2.43$ radians. So, $z = \sqrt{7}(\cos 2.43 + i\sin 2.43)$ hence $z^{-3} = \left[\sqrt{7}(\cos 2.43 + i\sin 2.43)\right]^{-3}$ which simplifies to

$$\begin{aligned} z^{-3} &= \left(\sqrt{7}\right)^{-3}(\cos 2.43 + i\sin 2.43)^{-3}, \\ &= \left(\sqrt{7}\right)^{-3}(\cos 7.29 - i\sin 7.29), \\ &= \left(\sqrt{7}\right)^{-3}(\cos 1 - i\sin 1). \end{aligned}$$

The next question is, Does this also work when n is a fraction? Yes. To prove this we set $n = p/q$ where p and q are positive integers. So now we are claiming that if $z = r(\cos\theta + i\sin\theta)$,

$$[r(\cos\theta + i.\sin\theta)]^{p/q} = r^{p/q}\left(\cos\frac{p\theta}{q} + i\sin\frac{p\theta}{q}\right),$$

is true.

Proof part 3: We apply the result of part 1 of the proof to the left hand side of the above expression to get

$$[r(\cos\theta + i.\sin\theta)]^{p/q} = r^{p/q}(\cos p\theta + i\sin p\theta)^{\frac{1}{q}}.$$

Since θ is a real number we can apply the usual rules of arithmetic to the right hand side of the above expression. In other words we raise both sides to the power q:

$$\left[[r(\cos\theta + i.\sin\theta)]^{\frac{p}{q}}\right]^q = \left(r^{\frac{p}{q}}\right)^q\left[(\cos p\theta + i\sin p\theta)^{\frac{1}{q}}\right]^q,$$

which simplifies to

$$[r(\cos\theta + i.\sin\theta)]^p = r^p(\cos p\theta + i\sin p\theta),$$

which is true by part 1 of the proof. Hence we have proved that

$$[r(\cos\theta + i.\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

is true for all $n \in \mathbb{Q}$. ■

For example, if $z = -\sqrt{2} - i$ then we can find $z^{1/2}$ by first converting to polar form and then using the above result. Hence $r = |z| = \sqrt{3}$ and $\theta = \arg(z) = \tan^{-1}\left(-1/-\sqrt{2}\right) - \pi \approx -2.53$. hence $z = \sqrt{3}(\cos(-2.53) + i\sin(-2.53))$ and

$$z^{1/2} = \left[\sqrt{3}(\cos(-2.53) + i\sin(-2.53))\right]^{1/2} = \sqrt[4]{3}\cos(-1.26) + i\sin(-1.26).$$

The final question is, Does this also work when n is a real (/irrational) number? In other words is it true that $[r(\cos\theta + i.\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$ when $n = \sqrt{2}$ or when $n = \pi$? Yes, but we will have to wait until part II of these notes when we deal with the exponential form of a complex number.

Forshadowing the fact that DeMoivre's theorem works for all real value of n we are now in a position to state DeMoivre's Theorem in full: if $z = r(\cos\theta + i\sin\theta)$ then

$$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta) \tag{39}$$

for all $n \in \mathbb{R}$.

Examples

1) Simplify the following complex numbers, expressing them in principal-argument form:

a) $z = \left[\sqrt{2}\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)\right]^{12}$ b) $z = \left(\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}\right)^{12}\left[2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right]^{5}$;

c) $z = \dfrac{\left[8\left(\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right)\right]^3}{\left[2\left(\cos\frac{\pi}{16} + i\sin\frac{\pi}{16}\right)\right]^{10}}$

Solutions

a) Using DeMoivre's theorem we have

$$\left[\sqrt{2}\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)\right]^{12} = \left(\sqrt{2}\right)^{12}\left(\cos\frac{12\pi}{8} + i\sin\frac{12\pi}{8}\right) = 64\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right).$$

Expressing this in principal-argument form we get

$$\left[\sqrt{2}\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)\right]^{12} = 64\left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right).$$

b) Using DeMoivre's theorem we have

$$\left(\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}\right)^{12}\left[2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right]^{5} = \left(\cos\frac{12\pi}{9} + i\sin\frac{12\pi}{9}\right) \times 2^5\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right).$$

Expressing this in principal-argument form we get

$$\left(\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}\right)^{12}\left[2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right]^{5} = 32\left(\cos\frac{2\pi}{3} - i\sin\frac{2\pi}{3}\right)\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right).$$

c) Using DeMoivre's theorem we have

$$z = \frac{\left[8\left(\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right)\right]^3}{\left[2\left(\cos\frac{\pi}{16} + i\sin\frac{\pi}{16}\right)\right]^{10}},$$

$$\text{So } z = \left[8\left(\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right)\right]^3 \left[2\left(\cos\frac{\pi}{16} + i\sin\frac{\pi}{16}\right)\right]^{-10},$$

$$= \frac{8^3}{2^{10}}\left(\cos\frac{9\pi}{8} + i\sin\frac{9\pi}{8}\right)\left(\cos\left(\frac{-10\pi}{16}\right) + i\sin\left(\frac{-10\pi}{16}\right)\right),$$

Expressing this in principal-argument form we obtain

$$z = \frac{1}{2}\left(\cos\frac{7\pi}{8} - i\sin\frac{7\pi}{8}\right)\left(\cos\left(\frac{5\pi}{8}\right) - i\sin\left(\frac{5\pi}{8}\right)\right).$$

2) The expression $\cos\theta + i\sin\theta$ is sometimes written as $\operatorname{cis}\theta$. As such, simplify

$$z = \frac{(\operatorname{cis}5\theta)^3(\operatorname{cis}\theta)^{-3}}{(\operatorname{cis}2\theta)^5(\operatorname{cis}3\theta)^2}.$$

<u>*Solution*</u>

From the numerator we have $(\operatorname{cis}5\theta)^3(\operatorname{cis}\theta)^{-3} = (\operatorname{cis}15\theta)(\operatorname{cis}(-3\theta)) = \operatorname{cis}12\theta$. From the denominator we have $(\operatorname{cis}2\theta)^5(\operatorname{cis}3\theta)^2 = (\operatorname{cis}10\theta)(\operatorname{cis}6\theta) = \operatorname{cis}16\theta$. Hence

$$\begin{aligned} z &= \frac{(\operatorname{cis}5\theta)^3(\operatorname{cis}\theta)^{-3}}{(\operatorname{cis}2\theta)^5(\operatorname{cis}3\theta)^2}, \\ &= \frac{\operatorname{cis}12\theta}{\operatorname{cis}16\theta}, \\ &= (\operatorname{cis}12\theta)(\operatorname{cis}16\theta)^{-1}, \\ &= (\operatorname{cis}12\theta)(\operatorname{cis}(-16\theta)) = \operatorname{cis}(-4\theta). \end{aligned}$$

Therefore

$$z = \cos 4\theta - i\sin 4\theta.$$

3) If $z = -\sqrt{3} - i$, express z^{15} in the form $x + iy$.

Solution: Given $z = -\sqrt{3} - i$ we have $r = |z| = \sqrt{\left(-\sqrt{3}\right)^2 + (-1)^2} = 2$, and $\arg(z) = \tan^{-1}\left((-1)/\left(-\sqrt{3}\right)\right) - \pi = -5\pi/6$. So

$$z = 2\left(\cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6}\right).$$

Hence

$$\begin{aligned} z^{15} &= 2^{15}\left(\cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6}\right)^{15}, \\ &= 2^{15}\left(\cos\frac{75\pi}{6} - i\sin\frac{75\pi}{6}\right), \\ &= 32768\left(\cos\frac{3\pi}{2} - i\sin\frac{3\pi}{2}\right), \\ &= 32768\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 32768i\,. \end{aligned}$$

4) Let $z = \frac{1}{2}\sqrt{3} + \frac{1}{2}i$. Find the least value of n, *where* $n \in \mathbb{N}$, which satisfies $z^n = -1$.

Solution: Given $z = \frac{1}{2}\sqrt{3} + \frac{1}{2}i, r = |z| = 1$ and $\theta = \operatorname{Arg}(z) = \pi/6$, from which we can write

$$z = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}.$$

Hence

$$z^n = \left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^n = \cos\frac{n\pi}{6} + i\sin\frac{n\pi}{6}.$$

We know that $\cos\pi = -1$ and $\sin\pi = 0$. Therefore, we want $\pi = n\pi/6$, implying $n = 6$. Hence

$$z^6 = \left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^6 = -1\,.$$

5) Let $z = -\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}\,i$. Find the least value of n, *where* $n \in \mathbb{Q}$, which satisfies $z^n = i$.

Solution: Given $z = -\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}\,i$, $r = |z| = 1$ and $\arg(z) = -\pi/4$. So $\theta = \operatorname{Arg}(z) = 3\pi/4$, From which we can write

$$z = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}.$$

Hence

$$z^n = \left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)^n = \cos\frac{3n\pi}{4} + i\sin\frac{3n\pi}{4}.$$

We know that $\cos(\pi/2) = 0$ and $\sin(\pi/2) = 1$. Therefore, we want $\pi/2 = 3n\pi/4$, implying $n = 2/3$.

Hence

$$z^{2/3} = \left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)^{2/3} = i\ .$$

6) Is it ever possible to have a complex number such as $z = \cos(\pi/4) + i.\sin(2\pi/3)$? Justify your answer.

Solution: Remember that the argument θ represents the angle the complex number makes with the positive real axis. There is only one unique angle for any given complex number, so the above expression in not a valid representation of a complex number.

7) Simplify $(\sin\theta + i\cos\theta)^3$.

Solution: Note that DeMoivre's theorem applies only to complex numbers of the form $\cos\theta + i\sin\theta$. The expression above is not is this form. However, we can transform it into a complex number as follows:

$$\begin{aligned}
(\sin\theta + i\cos\theta)^3 &= \left(\frac{-i}{-i}\right)^3 (\sin\theta + i\cos\theta)^3\ , \\
&= \left(-\frac{1}{i}\right)^3 (-i\sin\theta - i^2\cos\theta)^3\ , \\
&= -\frac{1}{i^3}(\cos\theta - i\sin\theta)^3\ ,
\end{aligned}$$

Now we can apply DeMoivre's theorem. Since $-1/i^3 = -i$ we obtain

$$(\sin\theta + i\cos\theta)^3 = -\sin 3\theta - i\cos 3\theta\ .$$

8) Simplify $(1 + \cos\theta + i\sin\theta)^3$.

Solution: Again note that DeMoivre's theorem applies only to $\cos\theta + i\sin\theta$. To transform the above expression into this form rewrite it as

$$\begin{aligned}
(1 + \cos\theta + i\sin\theta)^3 &= (\cos 0 + i\sin 0 + \cos\theta + i\sin\theta)^3\ , \\
&= \big(\cos 0 + \cos\theta + i(\sin 0 + \sin\theta)\big)^3\ ,
\end{aligned}$$

Now use the trig identitity called factor formula to get

$$
\begin{aligned}
(1+\cos\theta+i\sin\theta)^3 &= \left(2\cos\frac{\theta}{2}.\cos\frac{\theta}{2}+i\left(2\sin\frac{\theta}{2}.\cos\frac{\theta}{2}\right)\right)^3, \\
&= \left(2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)\right)^3, \\
&= 8\cos^3\frac{\theta}{2}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)^3, \\
&= 8\cos^3\frac{\theta}{2}\left(\cos\frac{3\theta}{2}+i\sin\frac{3\theta}{2}\right).
\end{aligned}
$$

9) Simplify $(1+\cos 2\theta+i\sin 2\theta)/(\cos 2\theta+i\sin 2\theta)$.

<u>*Solution:*</u> Again note that DeMoivre's theorem applies only to $\cos\theta+i\sin\theta$, so we first convert the numerator into such a form.

$$
\begin{aligned}
\frac{1+\cos 2\theta+i\sin 2\theta}{\cos 2\theta+i\sin 2\theta} &= \frac{\cos 0+i\sin 0+\cos 2\theta+i\sin 2\theta}{\cos 2\theta+i\sin 2\theta}, \\
&= \frac{\cos 0+\cos 2\theta+i(\sin 0+\sin 2\theta)}{\cos 2\theta+i\sin 2\theta}, \\
&= \frac{2\cos\theta\cos\theta+2i\sin\theta\cos\theta}{\cos 2\theta+i\sin 2\theta}, \\
&= \frac{2\cos\theta\,(\cos\theta+i\sin\theta)}{\cos 2\theta+i\sin 2\theta}, \\
&= 2\cos\theta\,(\cos\theta+i\sin\theta)(\cos 2\theta+i\sin 2\theta)^{-1}, \\
&= 2\cos\theta\,(\cos\theta+i\sin\theta)(\cos(-2\theta)+i\sin(-2\theta)), \\
&= 2\cos\theta\,(\cos\theta-i\sin\theta).
\end{aligned}
$$

<u>*Exercise:*</u> Simplify $(1-\sin\theta-i\cos\theta)^{-3}$.

<u>*Answer*</u>: $z=\frac{1}{8}\operatorname{cosec}^3\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\left(\cos\left(\frac{3\pi+6\theta}{4}\right)+i\sin\left(\frac{3\pi+6\theta}{4}\right)\right)$.

1.12.3 The periodicity of a complex number

We now know that $z^n = [r(\cos\theta + i.\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$ for all real values of n. However, in general we also know that $\cos\theta$ and $\sin\theta$ are periodic, so that $\cos\theta = \cos(\theta + 2k\pi)$ and $\sin\theta = \sin(\theta + 2k\pi)$. For example, if $z = \sqrt{3} + i$ then $z = 2(\cos\pi/6 + i\sin\pi/6)$. Cubing this value gives

$$z^3 = 2^3(\cos\pi/6 + i\sin\pi/6)^3 = 8(\cos\pi/2 + i\sin\pi/2) = 8i\,.$$

But it is also true that

$$z^3 = 8(\cos(\pi/2 + 2\pi) + i\sin(\pi/2 + 2\pi)) = 8(\cos 5\pi/2 + i\sin 5\pi/2) = 8i\,,$$

$$z^3 = 8(\cos(\pi/2 + 4\pi) + i\sin(\pi/2 + 4\pi)) = 8(\cos 9\pi/2 + i\sin 9\pi/2) = 8i\,,$$

etc. Such an aspect of periodicity will be of fundamental importance when we come to taking roots of complex number, i.e. performing $\sqrt[n]{z} = z^{1/n}$ (see later).

However, one important thing to note is the point at which we consider the periodicity of the complex number $z^n = r^n(\cos\theta + i\sin\theta)^n$, since there are two ways to look at it: either

i) we first apply Demoivre's theorem and then take account of the aspect of periodicity, i.e. $z^n = r^n(\cos\theta + i\sin\theta)^n$ leads to $z^n = \cos n\theta + i\sin n\theta$ by DeMoivre's theorem, and thence $z^n = \cos(n\theta + 2k\pi) + i\sin(n\theta + 2k\pi)$,

or

ii) we first take account of the aspect of periodicity, and then we apply DeMoivre's theorem, i.e. $z^n = r^n(\cos\theta + i\sin\theta)^n$ leads to $z^n = (\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi))^n$, and thence $z^n = \cos n(\theta + 2k\pi) + i\sin n(\theta + 2k\pi)$.

These two approaches are different and will not give the same results. We can see why this is so by considering the following example: letting $\mathrm{cis}(\theta) = \cos\theta + i\sin\theta$ we have seen above that if $z = 2\,\mathrm{cis}\,\pi/6$ then approach i) gives

$$z^3 = 8(\mathrm{cis}(\pi/6))^3 = 8\,\mathrm{cis}(3\pi/6) = 8\,\mathrm{cis}(\pi/2) = 8\,\mathrm{cis}(\pi/2 + 2k\pi)$$

leading to $z^3 = 8\,\mathrm{cis}\,\pi/2$ or $z^3 = 8\,\mathrm{cis}\,5\pi/2$ or $z^3 = 8\,\mathrm{cis}\,9\pi/2$, etc., for $k = 0, \pm1, \pm2, \pm3, \ldots$

However, approach ii) gives us

$$z^3 = 8(\mathrm{cis}(\pi/6 + 2k\pi))^3 = 8\,\mathrm{cis}\,3(\pi/6 + 2k\pi) = 8\,\mathrm{cis}(\pi/2 + 6k\pi)$$

leading to $z^3 = 8\,\mathrm{cis}\,\pi/2$ or $8\,\mathrm{cis}\,13\pi/2$ or $8\,\mathrm{cis}\,25\pi/2$ etc.

In comparing both sets of answer

$$z^3 = \{8 \text{ cis } \pi/2, 8 \text{ cis } 5\pi/2, 8 \text{ cis } 9\pi/2, 8 \text{ cis } 13\pi/2, 8 \text{ cis } 17\pi/2, 8 \text{ cis } 21\pi/2, 8 \text{ cis } 25\pi/2, \dots\}$$

and

$$z^3 = \{8 \text{ cis } \pi/2, 8 \text{ cis } 13\pi/2, 8 \text{ cis } 25\pi/2, 8 \text{ cis } 37\pi/2, 8 \text{ cis } 49\pi/2 \dots\}$$

it is clear that using approach ii) causes us to lose some answers which do indeed satify z^3. The reason for this is that, in approach ii), we end up multiplying the $2k\pi$ periodicity number by n, giving us z^3 values based on $2nk\pi$. This has the effect of skipping over intermediate values of z^3 based on $2k\pi$.

So, if we want to find *all possible answers* which satisfy z^3 we must adopt approach i), which in general is given by

$$z^n = r^n(\cos(n\theta + 2k\pi) + i\sin(n\theta + 2k\pi)) .$$

Understanding this aspect of when to take account of periodicity will be important when we come to proving properties of $\text{Arg}(z)$ as well as when taking roots of complex numbers.

1.12.4 Multiplication and division of complex numbers in polar form

We have seen that we can more easily perform exponentiation on complex numbers such as $z = (0.3487 - 6.149i)^{23}$ by first converting them to polar form, and then using DeMoivre's theorem. But what if we had $z = \left[(1+i)(2-3i)\left(-1-i\sqrt{3}\right)\right]/[(-2+3i)(4-i)]$? Here we would have to expand, simplify and use the conjugate of the denominator, all of which would be quite laborious.

However, there is a way of using the polar form of a complex number to perform multiplication and division in a very simple way. This is one of the advantages of converting a complex number into polar form as it drastically reduces the effort in the arithmetic of multiplication and division. And whereas multiplication and division of ever more factors of complex numbers in Cartesian form become more and more laborious, multiplication and division of complex numbers in polar form remains simple however many factors we have.

To see this consider $z_1 = \sqrt{3} - i$ and $z_2 = -2 + 2i$. To form their product we could multiply these in their current form, and this would be simple enough. But for the purpose of highlighting the aspect of arithmetic in polar form let us now convert these to be $z_1 = 2(\cos\pi/6 + i\sin\pi/6)$ and $z_2 = \sqrt{8}(\cos 3\pi/4 + i.\sin 3\pi/4)$.

Then

$$
\begin{aligned}
z_1 z_2 &= \left[2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right]\left[\sqrt{8}\left(\cos\frac{3\pi}{4} + i.\sin\frac{3\pi}{4}\right)\right],\\
&= \left(2\sqrt{8}\right)\left[\left(\cos\frac{\pi}{6}.\cos\frac{3\pi}{4} - \sin\frac{\pi}{6}.\sin\frac{3\pi}{4}\right) + i\left(\cos\frac{\pi}{6}.\sin\frac{3\pi}{4} + \sin\frac{\pi}{6}.\cos\frac{3\pi}{4}\right)\right],\\
&= \left(2\sqrt{8}\right)\left(\cos\left(\frac{\pi}{6} + \frac{3\pi}{4}\right) + i\sin\left(\frac{\pi}{6} + \frac{3\pi}{4}\right)\right), \qquad (*)\\
&= 2\sqrt{8}\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right).
\end{aligned}
$$

where expression (*) was obtained using the standard sum rules for trig identities. Now, it seems logical that, in forming the product $z_1 z_2$, we should multiply the separate moduli. But the significant and non-intuitive part of step (*) is that multiplication of two complex numbers in polar form implies the addition of their separate arguments. Is this just a coincidence? What if we were to multiply z_1, z_2 and z_3 where $z_3 = 1 + i\sqrt{3}$.

In polar form this latter complex number is $z_3 = 2(\cos\pi/3 + i\sin\pi/3)$, hence

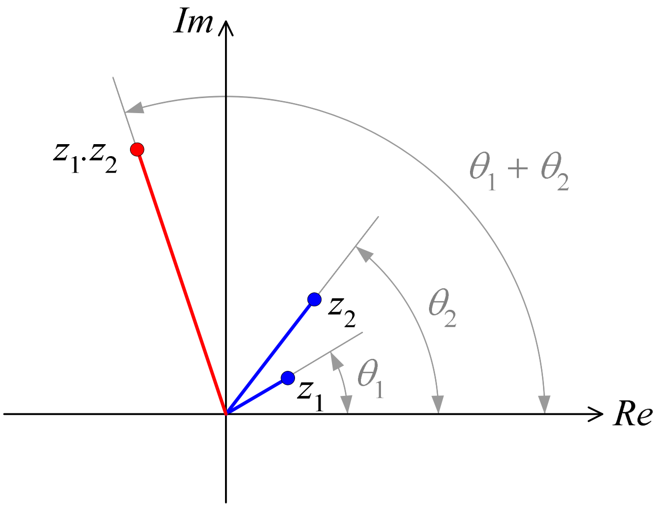
$$
\begin{aligned}
z_1 z_2 z_3 &= \left[2\sqrt{8}\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right)\right]\left[2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right],\\
&= \left(4\sqrt{8}\right)\left[\left(\cos\frac{11\pi}{12}.\cos\frac{\pi}{3} - \sin\frac{11\pi}{12}.\sin\frac{\pi}{3}\right)\right.\\
&\qquad\left. + i\left(\cos\frac{11\pi}{12}.\sin\frac{\pi}{3} + \sin\frac{11\pi}{12}.\cos\frac{\pi}{3}\right)\right],\\
&= \left(4\sqrt{8}\right)\left(\cos\left(\frac{11\pi}{12} + \frac{\pi}{3}\right) + i\sin\left(\frac{11\pi}{12} + \frac{\pi}{3}\right)\right),\\
&= 2\sqrt{8}\left(\cos\frac{45\pi}{36} + i\sin\frac{45\pi}{36}\right).
\end{aligned}
$$

So here we see that the product of three complex numbers is formed by the product of their moduli and the sum of their arguments. This should be obvious since we have seen that it works for $z_1 z_2$, and that $z_1 z_2 z_3 = (z_1 z_2) z_3 = z.z_3$ which is just the product of two complex numbers.

This therefore seems to suggest that, given two complex numbers $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ we have

$$z_1 z_2 = r_1.r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))\,,$$

as illustrated below, where the blue lines are of length r_1 and r_2 respectively, and the red line is of length $r_1 r_2$.



Note that the above expression is a generalisation of $z^2 = r^2(\cos 2\theta + i\sin 2\theta)$ since this latter can be expressed as $z^2 = r.r(\cos(\theta+\theta) + i\sin(\theta+\theta))$.

We can therefore say that

$\lvert z_1 z_2\rvert = \lvert z_1\rvert\lvert z_2\rvert$	and	$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$. to within multiples of 2π	(40)

So, just as for logs and exponentials, the process of "adding when multiplying" applies also to complex numbers.

<u>*Example 1:*</u> If $z_1 = \sqrt{3} + i$ and $z_2 = -1 + i$ then $|z_1| = 2$ and $|z_2| = \sqrt{2}$. Also, $\arg(z_1) = \tan^{-1}(1/\sqrt{3}) = \pi/6$ and $\arg(z_2) = \tan^{-1}(-1) = 3\pi/4$. Therefore

$$|z_1 z_2| = 2\sqrt{2} \text{ and } \arg(z_1) + \arg(z_2) = \pi/6 + 3\pi/4 = 11\pi/12.$$

<u>*Example 2:*</u> If $z_1 = i$ and $z_2 = -1 - i$ then $|z_1| = 1$ and $|z_2| = \sqrt{2}$. Also, $\arg(z_1) = \tan^{-1}(1/0) = \pi/2$ and $\arg(z_2) = \tan^{-1}(-1/-1) + \pi = 5\pi/4$ (remember this is "arg" not "Arg" so we are not taking the principal argument here). Hence

$$|z_1 z_2| = \sqrt{2} \text{ and } \arg(z_1) + \arg(z_2) = \pi/2 + 5\pi/4 = 7\pi/4.$$

Example 3: If $z_1 = -1 + i$ and $z_2 = -1 - i$ then $|z_1| = \sqrt{2}$ and $|z_2| = \sqrt{2}$. Also, $\arg(z_1) = \tan^{-1}(1/-1) + \pi = 3\pi/4$ and $\arg(z_2) = \tan^{-1}(-1/-1) + \pi = 5\pi/4$ (remember this is "arg" not "Arg" so we are not taking the principal argument here). Therefore

$$|z_1 z_2| = 2 \text{ and } \arg(z_1) + \arg(z_2) = 3\pi/4 + 5\pi/4 = 2\pi.$$

Example 4: If $z_1 = -1$ and $z_2 = -i$ then $|z_1| = 1$ and $|z_2| = 1$. Also, $\arg(z_1) = \tan^{-1}(0/-1) + \pi = \pi$ and $\arg(z_2) = \tan^{-1}(-1/0) + \pi = 3\pi/2$ (remember this is "arg" not "Arg" so we are not taking the principal argument here). Therefore

$$|z_1 z_2| = \text{and } \arg(z_1) + \arg(z_2) = \pi + 3\pi/2 = 5\pi/2 = \pi/2 + 2\pi.$$

Note that property (40) is not generally true for $\mathrm{Arg}(z)$. For example, if $z_1 = -1$ and $z_2 = 5i$, then $z_1 z_2 = -5i$. hence $\mathrm{Arg}(z_1) = \tan^{-1}\big(0/(-1)\big) + \pi = \pi$, and $\mathrm{Arg}(z_2) = \tan^{-1}(5/0) = \pi/2$, hence

$$\mathrm{Arg}(z_1) + \mathrm{Arg}(z_2) = \pi + \pi/2 = 3\pi/2, \text{ and } \mathrm{Arg}(z_1 z_2) = \tan^{-1}(-5/0) = -\pi/2,$$

so here

$$\mathrm{Arg}(z_1 z_2) \neq \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2).$$

These should make sense since $\mathrm{Arg}(z)$ is restricted to $(-\pi, \pi]$, hence adding or subracting any two angles, each in this interval, may easily result in an angle outside this interval. However, since there is no restriction on the size of the angle for $\arg(z)$ then the properties above hold.

What of divison? Let us again consider z_1 and z_2 as above, but this time perform z_1/z_2. Then

$$\begin{aligned}
z_1/z_2 &= \frac{2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)}{\sqrt{8}\left(\cos\frac{3\pi}{4} + i.\sin\frac{3\pi}{4}\right)}, \\
&= \frac{2}{\sqrt{8}}\left[\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\left(\cos\frac{3\pi}{4} + i.\sin\frac{3\pi}{4}\right)^{-1}\right], \\
&= \frac{2}{\sqrt{8}}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\left(\cos\frac{3\pi}{4} - i.\sin\frac{3\pi}{4}\right), \\
&= \frac{2}{\sqrt{8}}\left[\left(\cos\frac{\pi}{6}.\cos\frac{3\pi}{4} + \sin\frac{\pi}{6}.\sin\frac{3\pi}{4}\right) + i\left(\sin\frac{\pi}{6}.\cos\frac{3\pi}{4} - \cos\frac{\pi}{6}.\sin\frac{3\pi}{4}\right)\right],
\end{aligned}$$

$$= \frac{2}{\sqrt{8}}\left(\cos\left(\frac{\pi}{6}-\frac{3\pi}{4}\right)+i\sin\left(\frac{\pi}{6}-\frac{3\pi}{4}\right)\right),$$

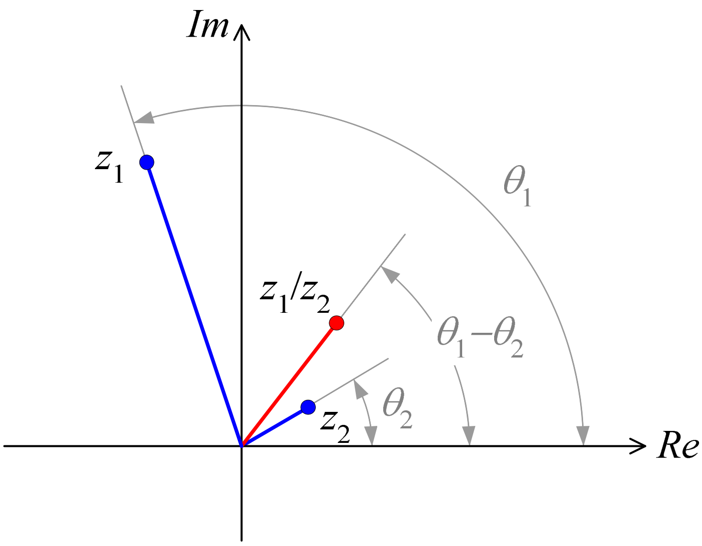
$$= 2\sqrt{8}\left(\cos\frac{7\pi}{12}-i\sin\frac{7\pi}{12}\right).$$

Again, it seems logical that, in forming the division z_1/z_2, we should divide the separate moduli. But the significant part is that division of two complex numbers in polar form implies the subtraction of their separate arguments. And again this is not a coincidence, but is a general property of the division of complex numbers in polar form.

Therefore given two complex numbers $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ we have

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)),$$

which is illustrated below, and where the blue lines are of length r_1 and r_2 respectively, and the red line is of length r_1/r_2.



We can therefore say that

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \qquad \text{and} \qquad \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)\,. \tag{41}$$

What all of this means is that we can perform the arithmetic of multiplication or division of any number of factors of complex numbers in polar form with very little effort compared to performing this arithmetic in Cartesian form.

Example 5: If $z_1 = \sqrt{3} + i$ and $z_2 = -1 + i$ then $|z_1| = 2$ and $|z_2| = \sqrt{2}$. Also, $\arg(z_1) = \tan^{-1}(1/\sqrt{3}) = \pi/6$ and $\arg(z_2) = \tan^{-1}(-1) = 3\pi/4$. Therefore

$$|z_1/z_2| = 2/\sqrt{2} \text{ and } \arg(z_1) - \arg(z_2) = \pi/6 - 3\pi/4 = -7\pi/12.$$

Example 6: If $z_1 = i$ and $z_2 = -1 - i$ then $|z_1| = 1$ and $|z_2| = \sqrt{2}$. Also, $\arg(z_1) = \tan^{-1}(1/0) = \pi/2$ and $\arg(z_2) = \tan^{-1}(-1/-1) + \pi = 5\pi/4$ (remember this is "arg" not "Arg" so we are not taking the principal argument here). Hence

$$|z_1/z_2| = 1/\sqrt{2} \text{ and } \arg(z_1) - \arg(z_2) = \pi/2 - 5\pi/4 = -3\pi/4.$$

Example 7: If $z_1 = -1 + i$ and $z_2 = -1 - i$ then $|z_1| = \sqrt{2}$ and $|z_2| = \sqrt{2}$. Also, $\arg(z_1) = \tan^{-1}(1/-1) + \pi = 3\pi/4$ and $\arg(z_2) = \tan^{-1}(-1/-1) + \pi = 5\pi/4$ (remember this is "arg" not "Arg" so we are not taking the principal argument here). Therefore

$$|z_1/z_2| = 1 \text{ and } \arg(z_1) - \arg(z_2) = 3\pi/45\pi/4 = -\pi/2.$$

Note that property (41) is not generally true for $\mathrm{Arg}(z)$. For example, if $z_1 = -1$ and $z_2 = 5i$, then $z_1/z_2 = i/5$. So $\mathrm{Arg}(z_1) = \tan^{-1}\big(0/(-1)\big) + \pi = \pi$, and $\mathrm{Arg}(z_2) = \tan^{-1}(5/0) = \pi/2$, hence

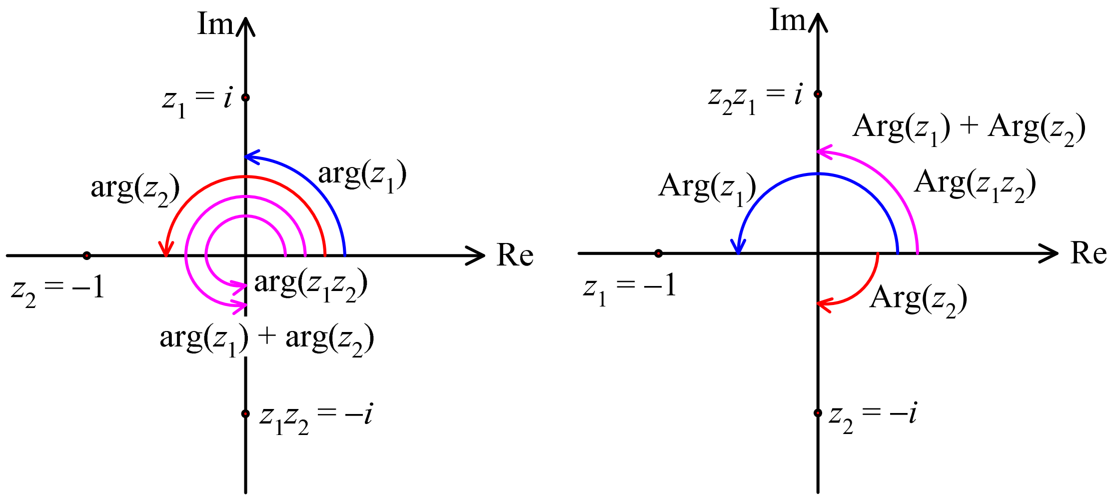
$$\mathrm{Arg}(z_1) - \mathrm{Arg}(z_2) = \pi - \pi/2 = \pi/2, \text{ and } \mathrm{Arg}(z_1/z_2) = \tan^{-1}(0.2/0) = \pi/2,$$

so here $$\mathrm{Arg}(z_1/z_2) \neq \mathrm{Arg}(z_1) - \mathrm{Arg}(z_2).$$

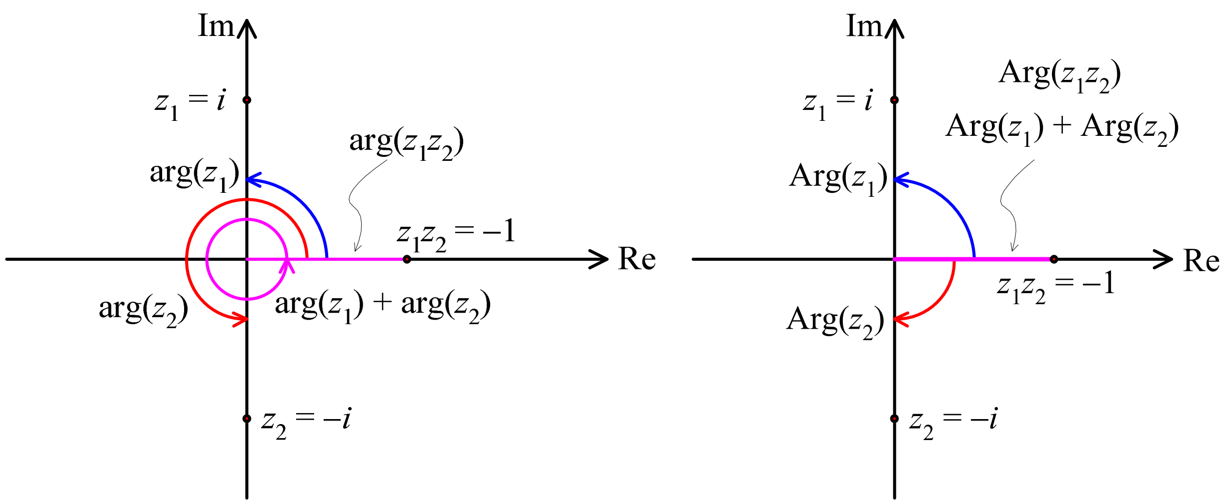
The diagrams on the next page illustrate the multiplication/addition property of the arguments of two complex numbers. The blue arc represents $\arg(z_1)$, the red arc represents $\arg(z_2)$ and the pink arc represents both $\arg(z_1 z_2)$ and $\arg(z_1) + \arg(z_2)$.

Example 8: As another example suppose we are given that $z_1 = 2(\cos(\pi/8) + i.\sin(\pi/3))$ and $z_2 = 4(\cos(3\pi/8) + i.\sin(3\pi/3))$, and we want to find $z_1 z_2$. We then proceed as follows:

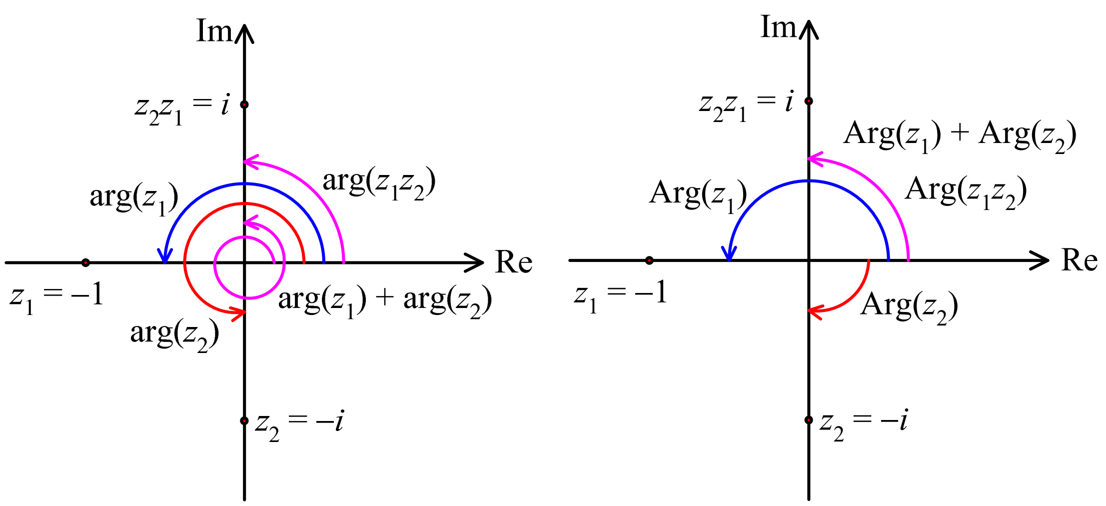
$$\begin{aligned}
z_1 z_2 &= [2(\cos(\pi/8) + i.\sin(\pi/3))][4(\cos(3\pi/8) + i.\sin(3\pi/3))]\,, \\
&= 8\left(\cos\left(\frac{\pi}{8} + \frac{3\pi}{8}\right) + i\sin\left(\frac{\pi}{8} + \frac{3\pi}{8}\right)\right), \\
&= 8\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right), \\
&= 8i\,.
\end{aligned}$$



The multiplication of –1 and i (non-principal argument version, and principal argument version)



The multiplication of –i and i (non-principal argument version and principal argument version)



The multiplication of –1 and –i (non-principal argument version and (principal argument version)

Example 9: Division presents no more difficulty. For example, suppose we want to simplify

a) $z = \dfrac{\cos 2\theta + i \sin 2\theta}{\cos\theta + i\sin\theta}$ b) $z = \dfrac{1}{\cos(-\theta) + i\sin(-\theta)}$

In this case we can do as follows :

$$
\begin{aligned}
\text{a)} \quad z &= \frac{\cos 2\theta + i \sin 2\theta}{\cos\theta + i\sin\theta}, \\
&= (\cos 2\theta + i\sin 2\theta)(\cos\theta + i\sin\theta)^{-1}, \\
&= (\cos 2\theta + i\sin 2\theta)(\cos(-\theta) + i\sin(-\theta)), \\
&= \cos\theta + i\sin\theta .
\end{aligned}
$$

and

$$
\begin{aligned}
\text{b)} \quad z &= \frac{1}{\cos(-\theta) + i\sin(-\theta)}, \\
&= (\cos(-\theta) + i\sin(-\theta))^{-1}, \\
&= \cos\theta + i\sin\theta .
\end{aligned}
$$

Example 10: Products of complex numbers involving ever high powers also present no real difficulty. In wanting to evaluate $(\cos(\pi/9) + i\sin(\pi/9))^{12} \times [2(\cos(\pi/6) + i\sin(\pi/6))]^5$ we have

$$
\begin{aligned}
&(\cos(\pi/9) + i\sin(\pi/9))^{12} \times [2(\cos(\pi/6) + i\sin(\pi/6))]^5 \\
&\quad = 32(\cos(12\pi/9) + i\sin(12\pi/9))\,(\cos(5\pi/6) + i\sin(5\pi/6)), \\
&\quad = 32\left(\cos\left(\frac{12\pi}{9} + \frac{5\pi}{6}\right) + i\sin\left(\frac{12\pi}{9} + \frac{5\pi}{6}\right)\right), \\
&\quad = 32\left(\cos\left(\frac{13\pi}{6}\right) + i\sin\left(\frac{13\pi}{6}\right)\right), \\
&\quad = 32\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right).
\end{aligned}
$$

In wanting to evaluate $[8(\cos(3\pi/8) + i\sin(3\pi/8))]^3 \div [2(\cos(\pi/16) + i\sin(\pi/16))]^{10}$ we have

$[8(\cos(3\pi/8) + i\sin(3\pi/8))]^3 \div [2(\cos(\pi/16) + i\sin(\pi/16))]^{10}$

$$= \frac{8^3}{2^{10}}\left(\cos\left(\frac{9\pi}{8}\right) + i\sin\left(\frac{9\pi}{8}\right)\right)\left(\cos\left(\frac{10\pi}{16}\right) + i\sin\left(\frac{10\pi}{16}\right)\right),$$

$$= \frac{1}{2}\left(\cos\left(\frac{9\pi}{8} + \frac{10\pi}{16}\right) + i\sin\left(\frac{9\pi}{8} + \frac{10\pi}{16}\right)\right),$$

$$= \frac{1}{2}\left(\cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right)\right),$$

$$= \frac{1}{2}\left(\cos\left(\frac{\pi}{4}\right) - i\sin\left(\frac{\pi}{4}\right)\right).$$

Example 11: Let $z = x + iy$. To show that $\arg(z.z^*) = 0$ we do the following:

$$\arg(z.z^*) = \arg\big((x+iy)(x-iy)\big) = \arg(x^2 + y^2).$$

Note that $x^2 + y^2$ is a positive real number. Since all positive real numbers lie on the *Re* axis, their angle w.r.t. to this axis is 0 rads. Hence $\arg(z.z^*) = 0$.

The algebraic way of seeing this is to note that we are looking for $\arg\big((x^2 + y^2) + 0.i\big)$. Therefore

$$\theta = \tan^{-1}\left(\frac{0}{x^2 + y^2}\right) = 0\,.$$

Exercise: Show that $\arg(z + z^*) = 0$.

Example 12: In order to evaluate $\left(1 + i\sqrt{3}\right)^8 + \left(1 - i\sqrt{3}\right)^8$ we use DeMoivre's theorem as follows: Let $z_1 = \left(1 + i\sqrt{3}\right)^8$. Then $r = |z_1| = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2$, and $\theta = \arg(z) = \tan^{-1}\sqrt{3} = \pi/3$. Hence

$$z_1 = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right).$$

Similarly we can show that

$$z_2 = 2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right).$$

Hence

$$
\begin{aligned}
z_1^8 + z_2^8 &= \left[2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]^8 + \left[2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)\right]^8, \\
&= 2^8\left(\cos\frac{8\pi}{3} + i\sin\frac{8\pi}{3}\right) + 2^8\left(\cos\frac{8\pi}{3} - i\sin\frac{8\pi}{3}\right), \\
&= 2^8\left(2\cos\frac{8\pi}{3}\right), \\
&= -256.
\end{aligned}
$$

Exercise: Show that $(1+i)^n + (1-i)^n = 2\sqrt{2^n}.\cos(n\pi/4)$, where $n \in \mathbb{N}$. Deduce a similar result for $(1+i)^n - (1-i)^n$.

Exercise: Let $z = (a+bi)^n(a-ib)^n$, where $n \in \mathbb{N}$. Show that z is always real, and that $(a-ib)^n$ is the conjugate of $(a+ib)^n$.

Example 13: Suppose, more generally, that we want to find the value of θ such that

$$\cos(r\theta) + i\sin(r\theta) = \cos(s\theta) + i\sin(s\theta),$$

where r and s are integers and where$r \neq s$. We can do this by firstly dividing the LHS by the RHS, and then applying DeMoivre's theorem appropriately. So

$$
\begin{aligned}
\frac{\cos(r\theta) + i\sin(r\theta)}{\cos(s\theta) + i\sin(s\theta)} &= 1 \\
\Rightarrow \quad (\cos(r\theta) + i\sin(r\theta))(\cos(s\theta) + i\sin(s\theta))^{-1} &= 1 \\
\Rightarrow \quad (\cos(r\theta) + i\sin(r\theta))(\cos(-s\theta) + i\sin(-s\theta)) &= 1
\end{aligned}
$$

which by demoivre's theorem gives us

$$\cos(r-s)\theta + i\sin(r-s)\theta = 1.$$

Equating *Re* and *Im* parts we have $\cos(r-s)\theta = 1$ and $\sin(r-s)\theta = 0$. Both of these solve to give the same answer of $(r-s)\theta = 2n\pi$, for $n = 0, \pm1, \pm2, \ldots$ Hence

$$\theta = \frac{2n\pi}{r-s},$$

for $n = 0, \pm 1, \pm 2, \ldots$ In other words, we can find the general solution to $\cos\theta + i\sin\theta = \cos 2\theta + i\sin 2\theta$, $\cos 7\theta + i\sin 7\theta = \cos 3\theta - i\sin 3\theta$, or any other combination, by the use of the simple expression for θ above.

Other examples:

1) Evaluate

$$\frac{\sqrt{(\cos(\pi/6) - i\sin(\pi/6))^{11}}}{\sqrt{\cos(\pi/6) + i\sin(\pi/6\,)}}\,.$$

Solution:

$$\frac{\sqrt{(\cos(\pi/6) - i\sin(\pi/6))^{11}}}{\sqrt{\cos(\pi/6) + i\sin(\pi/6\,)}} = \frac{(\cos(\pi/6) - i\sin(\pi/6))^{11/2}}{(\cos(\pi/6) + i\sin(\pi/6\,))^{1/2}}\,.$$

Division of complex numbers implies subtraction of their arguments. However the numerator is not in the standard form $\cos\theta + i\sin\theta$, so we first have to convert the numerator into this form:

$$\frac{\sqrt{(\cos(\pi/6) - i\sin(\pi/6))^{11}}}{\sqrt{\cos(\pi/6) + i\sin(\pi/6\,)}} = \frac{(\cos(-\pi/6) + i\sin(-\pi/6))^{11/2}}{(\cos(\pi/6) + i\sin(\pi/6\,))^{1/2}}\,.$$

We can now apply DeMoivre's theorem and then perform the division by subtracting the arguments appropriately:

$$\begin{aligned}
\frac{\sqrt{(\cos(\pi/6) - i\sin(\pi/6))^{11}}}{\cos(\pi/6) + i\sin(\pi/6\,)} &= \frac{\cos(-11\pi/12) + i\sin(-11\pi/12)}{\cos(\pi/12) + i\sin(\pi/12\,)}\,, \\
&= \cos\left(-\frac{11\pi}{12} - \frac{\pi}{12}\right) + i\sin\left(-\frac{11\pi}{12} - \frac{\pi}{12}\right), \\
&= \cos(-\pi) + i\sin(-\pi)\,, \\
&= -1.
\end{aligned}$$

2) Let $\cos\theta + i.\sin\theta \equiv \operatorname{cis}\theta$. Find $(\operatorname{cis}A)(\operatorname{cis}B)(\operatorname{cis}C)$ when $A + B + C = \pi$. If $\pi < A + B + C \le 2\pi$, express your answer in principal-argument form.

Solution: $(\operatorname{cis}A)(\operatorname{cis}B)(\operatorname{cis}C) = (\operatorname{cis}(A+B))(\operatorname{cis}C) = (\operatorname{cis}(A+B+C)) = \operatorname{cis}\pi = -1$.

If $\pi < A + B + C \le 2\pi$ then $A + B + C$ is greater than the maximum allowed postive angle of π. Therefore we need to subtract 2π from the argument in order to bring this back into the interval $(-\pi, \pi]$. Hence the principal-argument form is

$$(\operatorname{cis}A)(\operatorname{cis}B)(\operatorname{cis}C) = \operatorname{cis}\big((A + B + C) - 2\pi\big)\,.$$

3) If $u = \cos\theta + i\sin\theta$ and $v = \cos\phi + i\sin\phi$, where $\theta \neq \phi$, find $(u/v) \pm (v/u)$.

Solution:

Since $u/v = (\cos\theta + i\sin\theta)/(\cos\phi + i\sin\phi) = (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)^{-1}$ and $v/u = (\cos\phi + i\sin\phi)/(\cos\theta + i\sin\theta) = (\cos\phi + i\sin\phi)(\cos\theta + i\sin\theta)^{-1}$ we have

$$\begin{aligned}\frac{u}{v}+\frac{v}{u} &= (\cos\theta + i\sin\theta)(\cos(-\phi) + i\sin(-\phi)) \\ &\qquad +(\cos\phi + i\sin\phi)(\cos(-\theta) + i\sin(-\theta)) \\ &= (\cos(\theta-\phi) + i\sin(\theta-\phi)) + (\cos(\phi-\theta) + i\sin(\phi-\theta))\,, \\ &= \cos(\theta-\phi) + \cos(\phi-\theta) + i(\sin(\theta-\phi) + \sin(\phi-\theta))\,.\end{aligned}$$

This last expression can be simplified using the factor formula of the trig family of identities to give us

$$\frac{u}{v}+\frac{v}{u} = 2\cos(\theta-\phi)\,.$$

By the same process it can be shown that

$$\frac{u}{v}-\frac{v}{u} = 2i\sin(\theta-\phi)\,.$$

4) Let $z = \cos\theta + i\sin\theta$. Simplifying $z + 1/z$ we get

$$\begin{aligned}z+\frac{1}{z} &= \cos\theta + i\sin\theta + \frac{1}{\cos\theta + i\sin\theta}\,, \\ &= \cos\theta + i\sin\theta + (\cos\theta + i\sin\theta)^{-1}\,, \\ &= \cos\theta + i\sin\theta + \cos\theta - i\sin\theta\,, \\ &= 2\cos\theta\,.\end{aligned}$$

By the same process it can be shown that $z - 1/z = 2i\sin\theta$.

Note that the simplicity in arithmetic applies only to performing multiplication and division, not to addition and subtraction. For example, given three complex numbers $z_1 = -1 - i$, $z_2 = 2 - i$ and $z_3 = -\sqrt{3} - i$ it is far easier to add and subtract these in the current form rather than adding /subtracting them in the form $z_1 = \cos 3\pi/4 + i\sin 3\pi/4$, $z_2 = \sqrt{5}(\cos 0.464 - i\sin 0.464)$ and $z_3 = 2(\cos 5\pi/6 - i\sin 5\pi/6)$.

5) To show that

$$\frac{2}{1+z} = 1 - i\tan\frac{\theta}{2},$$

we first look to convert the denominator into the form $\cos\theta + i\sin\theta$. Hence

$$\begin{aligned}
1+z &= 1+\cos\theta + i\sin\theta\ , \\
&= \cos 0 + i\sin 0 + \cos\theta + i\sin\theta\ , \\
&= \cos 0 + \cos\theta + i(\sin 0 + \sin\theta)\ , \\
&= 2\cos^2\left(\frac{\theta}{2}\right) + 2i\sin\frac{\theta}{2}.\cos\frac{\theta}{2},
\end{aligned}$$

where this last equation was derived using the factor formula of trig identities. So we have

$$\begin{aligned}
\frac{2}{1+z} &= \frac{2}{2\cos^2\left(\frac{\theta}{2}\right) + 2i\sin\frac{\theta}{2}.\cos\frac{\theta}{2}} \\
&= \frac{1}{\cos^2\left(\frac{\theta}{2}\right) + i\sin\frac{\theta}{2}.\cos\frac{\theta}{2}} \\
&= \frac{1}{\cos^2\left(\frac{\theta}{2}\right) + i\sin\frac{\theta}{2}.\cos\frac{\theta}{2}} \times \frac{\cos^2\left(\frac{\theta}{2}\right) - i\sin\frac{\theta}{2}.\cos\frac{\theta}{2}}{\cos^2\left(\frac{\theta}{2}\right) - i\sin\frac{\theta}{2}.\cos\frac{\theta}{2}} \\
&= \frac{\cos^2\left(\frac{\theta}{2}\right) - i\sin\frac{\theta}{2}.\cos\frac{\theta}{2}}{\cos^4\left(\frac{\theta}{2}\right) + \sin^2\frac{\theta}{2}.\cos^2\frac{\theta}{2}}\ .
\end{aligned}$$

Dividing top and bottom of the RHS by $\cos^2(\theta/2)$ gives $2/(1+z) = 1 - i\tan(\theta/2)$.

<u>*Exercise:*</u> If

$$z = \frac{1-\cos\theta + i\sin\theta}{1+\cos\theta - i\sin\theta},$$

Show that $Re(z) = 0$ and $Im(z) = \tan(\theta/2)$.

6) Suppose we want to solve for x in $x^{2n} - 2x^n \cos n\alpha + 1 = 0$. This looks like a quadratic in x^n so an obvious thing to do is to use the quadratic formula:

$$
\begin{aligned}
x^n &= \frac{2\cos n\alpha \pm \sqrt{4\cos^2 n\alpha - 4}}{2}, \\
&= \cos n\alpha \pm \sqrt{\cos^2 n\alpha - 1}, \\
&= \cos n\alpha \pm \sqrt{\sin^2 n\alpha}.
\end{aligned}
$$

Now notice that $-1 \leq \sin n\alpha \leq 1$ hence the maximum value that $\sin^2 n\alpha$ can reach is +1. Therefore, in general, the determinant Δ will be negative, with $\Delta = 0$ on specific occasions.

Hence we can write the above as

$$
\begin{aligned}
x^n &= \cos n\alpha \pm i \sin n\alpha, \\
&= \cos(\pm n\alpha) + i \sin(\pm n\alpha).
\end{aligned}
$$

Hence

$$
\begin{aligned}
x^n &= (\cos(\pm n\alpha) + i \sin(\pm n\alpha))^{1/n}. \\
&= (\cos(2k\pi \pm n\alpha) + i \sin(2k\pi \pm n\alpha))^{1/n}, \\
&= \cos\left(\frac{2k\pi}{n} \pm \alpha\right) + i \sin\left(\frac{2k\pi}{n} \pm \alpha\right).
\end{aligned}
$$

7) Find all complex numbers z such that $|z| = 1$ and

$$\left|\frac{z}{\bar{z}} + \frac{\bar{z}}{z}\right| = 1.$$

Solution:

Let $z = \cos\theta + i\sin\theta$. Then

$$1 = \left|\frac{z}{\bar{z}} + \frac{\bar{z}}{z}\right| = \frac{|z^2 + \bar{z}^2|}{|z|^2} = |\cos 2\theta + i \sin 2\theta + \cos\theta - i \sin 2\theta| = 2|\cos 2\theta|.$$

Therefore

$$\cos 2\theta = \frac{1}{2} \qquad \text{or} \qquad \cos 2\theta = -\frac{1}{2}$$

For

- $\cos 2\theta = \frac{1}{2}$ we have $\theta_1 = -5\pi/6, \theta_2 = -\pi/6, \theta_3 = \pi/6, \theta_4 = 5\pi/6$

- $\cos 2\theta = -\frac{1}{2}$ we have $\theta_5 = -2\pi/3, \theta_6 = -\pi/3, \theta_7 = \pi/3, \theta_8 = 2\pi/3$

Hence there are eight complex numbers z such that $|z| = 1$: $z_k = \cos\theta_k + i\sin\theta_k$ where θ_k are the arguments listed above for $k = 1$ to 8.

8) Let two complex numbers z_1, z_2 be such that $|z_1| = |z_2| = 1$. If $\arg(z_1/z_2) = \pi/2$, find

$$\left|\frac{z_1 + z_2}{z_1 - z_2}\right|.$$

<u>*Solution:*</u>

Let $z = \cos\theta + i\sin\theta$. Then

$$\begin{aligned}
\left|\frac{z_1 + z_2}{z_1 - z_2}\right| &= \left|\frac{(\cos\theta_1 + i\sin\theta_1) + (\cos\theta_2 + i\sin\theta_2)}{(\cos\theta_1 + i\sin\theta_1) + (\cos\theta_2 - i\sin\theta_2)}\right|, \\
&= \left|\frac{(\cos\theta_1 + \cos\theta_2) + i(\sin\theta_1 + \sin\theta_2)}{(\cos\theta_1 + \cos\theta_2) + i(\sin\theta_1 - \sin\theta_2)}\right|, \\
&= \frac{\cos^2\theta_1 + \sin^2\theta_1 + 2(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + \cos^2\theta_2 + \sin^2\theta_2}{\cos^2\theta_1 + \sin^2\theta_1 + 2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + \cos^2\theta_2 + \sin^2\theta_2}, \\
&= \frac{2 + 2\cos(\theta_1 - \theta_2)}{2 + 2\cos(\theta_1 + \theta_2)}.
\end{aligned}$$

Since $\arg(z_1/z_2) = \arg(z_1) - \arg(z_2) = \theta_1 - \theta_2 = \pi/2$, and therefore $\theta_2 = \theta_1 - \pi/2$, this last equation simplifies to

$$\left|\frac{z_1 + z_2}{z_1 - z_2}\right| = \frac{2}{2 + 2\cos(2\theta_1 - \pi/2)} = \frac{1}{1 + \sin(2\theta_1)}.$$

We would have got the same result if we had substitute for $\theta_1 = \theta_2 + \pi/2$, but this time with the denominator having term $\sin(2\theta_2)$.

1.12.5 The geometric effect of multiplication and division of complex numbers in polar form

We saw in sections (1.7.2) and (1.8.2) the geometric effect of multiplication and division on a complex numbers. This involved a number of transformations, including considering a triangle formed for one complex number, rotating the triangle so that its base met the hypoteneuse of the triangle formed for the other complex number, and scaling the base of the former triangle appropriately. All of this seems quite involved.

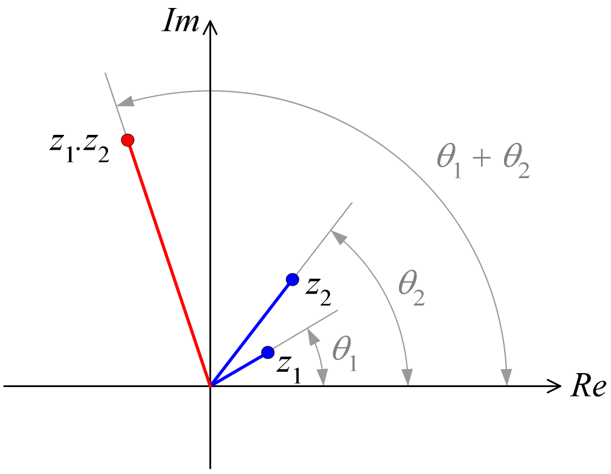
Having learnt about DeMoivre's theorem we are now in a position to be able to more easily see and understand the geometric effect of multiplication and division of complex numbers. So, given two complex numbers $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ we have

$$z_1 z_2 = r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)),$$

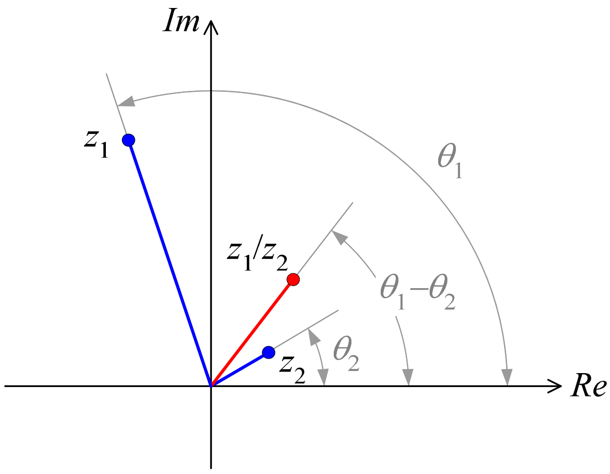
and

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos\theta_1 + i\sin\theta_1) \div (\cos\theta_2 + i\sin\theta_2) = \frac{r_1}{r_2}(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)).$$

From these expressions we can see directly the scaling and rotation effect of multiplication and division, namely that the scaling of one complex number by the other is automatically shown as $r_1 r_2$ or r_1/r_2, and the rotation of one complex number (already at a given angle) by the other is automatically shown as $\theta_1+\theta_2$ or as $\theta_1-\theta_2$. This situation is illustrated below.



The geometric effect of multiplication



The geometric effect of division